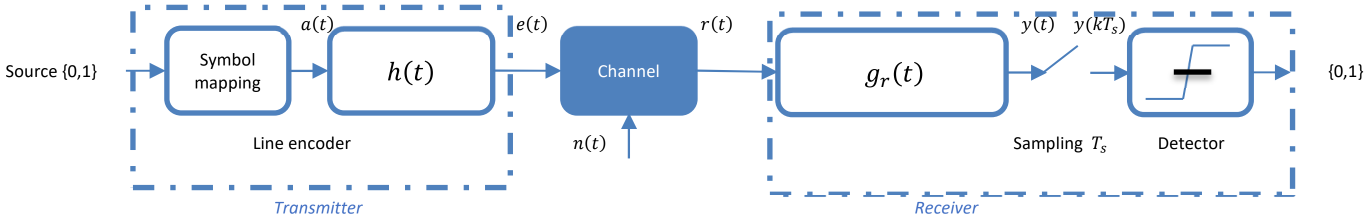
20

**Exercice 1**

Consider the following digital transmission scheme:

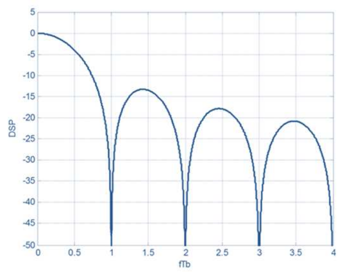



A] Assume the channel is ideal and the noise PSD is $S_n(f) = \sigma_n^2 = \frac{N_0}{2}$ **/ 6 points**

Assume that $h(t) = \prod_{T_s}(t - T_s/2)$ where $T_s = 0.1\ ms$

And $a(t) = \sum_k a_m\, \delta(t - kT_s)$ where $a_m = 1$ for bit=1 and $a_m = -1$ for bit=0

1. What is the role of $h(t)$
2. Express $e(t)$ and plot approximately its Power Spectral Density (PSD) (no calculation required).
3. Express $r(t)$.
4. What is the role of $g_r(t)$
5. Why do we perform sampling?
6. Calculate the optimal decision threshold and the express the BER (Bit Error Rate).

| | |
|---|---|
| **1.** h(t) = transmit pulse-shaping filter → converts Dirac impulses into rectangular pulses of duration Ts. | 0.5 pt |
| **2.** $e(t) = \sum_m a_m\ h(t - mT_s)$<br>PSD: $S_e(f) = T_s \cdot \text{sinc}^2(fT_s)$, null at $f = k/T_s = 10$kHz.<br><br> | 0.5 pt<br>1 pt |
| **3.** $r(t) = e(t) + n(t) = \sum_m a_m\ h(t - mT_s) + n(t)$ | 0.5 pt |
| **4.** $g_r(t)$= matched filter to h(t) → maximizes SNR at sampling instant $t_0$. | 0.5 pt |
| **5.** After the shaping filter, each symbol has a duration of $T_s$ , sampling is used to recover one unique symbol per $T_s$ | 1 pt |
| **6.** Optimal threshold (equiprobable symbols): $S_{opt} = 0$ and $E_h = T_s = E_b$ → BER: $P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$ | 1.5 pt |

B] Assume the channel is band-limited to bandwidth $B$, and $S_n(f) = \sigma_n^2 = \frac{N_0}{2}$ **/ 6 points**

1. Express the frequency response of the channel considered, assuming no distortion.
2. Express $r(t)$ again. What does its waveform become? What is the consequence on detection?

We choose to replace $h(t)$ by the following expression : $\frac{4\frac{t}{T_s}\cos\left(2\pi\frac{t}{T_s}\right)}{\pi\frac{t}{T_s}\left[1-\left(4\frac{t}{T_s}\right)^2\right]}$

3. What is the name of this filter? Give the expression of its Fourier Transform $H(f)$.
4. What is the purpose of this operation?
5. What does the eye diagram look like in this case?
6. Determine the spectral efficiency. Modify $h(t)$ so it can be doubled? What is the consequence?

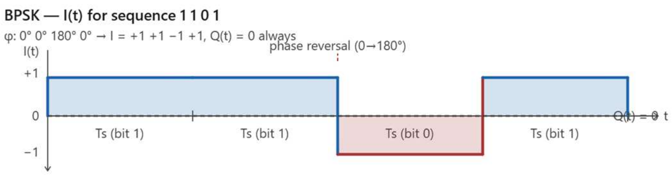
| | |
|---|---|
| **1.** Ideal lowpass channel of bandwidth B: $C(f) = \begin{cases} C_0 e^{-2\pi j f_0 t} & \lvert f\rvert < B \\ 0 & ailleurs \end{cases}$ | 0.5 pt |
| **2.** $r(t) = e(t) * c(t) + n(t)$<br>rectangular pulses spread in time → ISI (Inter-Symbol Interference) → eye diagram closes → detection errors increase. | 0.5 pt<br>1.5 pt |
| **3.** Filter name: Root Raised Cosine (RRC) with roll-off $\alpha = 1$.<br>$$H_{RRC}(f) = \sqrt{T_s}\cos\left(\frac{\pi T_s}{2}\lvert f\rvert\right),\ \lvert f\rvert \le \frac{1}{T_s}$$ | 0.5 pt<br>0.5 pt |
| **4.** RRC (Tx) × RRC (Rx) = overall RC → satisfies Nyquist ISI-free criterion: zero ISI at sampling instants, spectrum confined to $B = (1+\alpha)/(2T_s)$. | 1.5 pt |
| **5.** Eye diagram is fully open at sampling instants, with smooth transitions. | 0.5 pt |
| **6.**<br>• Spectral efficiency: $\eta = \frac{2\log_2 M}{1+\alpha}$ = 1bit/s/Hz (M=2, α=1)<br>• Reduce α → 0 (ideal sinc filter, unrealizable in practice).<br>• Eye diagram → open diamond at the center, but with very slowly decaying tails that cause multiple overlapping traces away from the center | 0.5 pt<br>0.5 pt<br>0.5 pt |

**Exercice 2** **/ 8 points**

We wish to compare two modulation techniques: **BPSK** and **BFSK**.

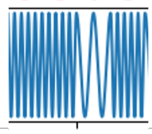
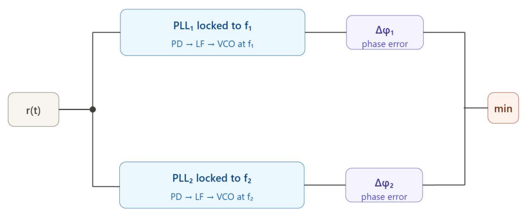
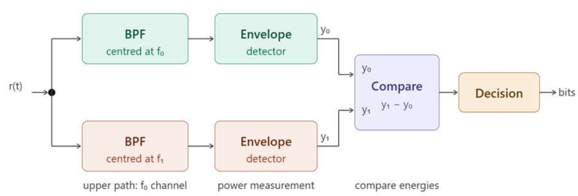
A] BPSK

1. Express the transmitted signal for BPSK, as well as its in-phase component I(t) and quadrature component Q(t).
2. Plot them for the following binary sequence: **1 1 0 1**
3. We note the presence of a 45° phase offset at the receiver — what happens ? And what about a 90° phase offset?
4. How can this be corrected? Explain.
5. Assume carrier recovery is imperfect and a residual frequency offset $\Delta f$ remain. What happens to the constellation?

<table>
<tr><td>A1. $m(t) = \overbrace{\sum_k \cos\varphi_k h_e(t-kT_s)}^{I(t)} A\cos(2\pi f_c t) - \overbrace{\sum_k \sin\varphi_k h_e(t-kT_s)}^{Q(t)} A\sin(2\pi f_c t)$<br>$I(t) = \sum_k \pm 1\, h_e(t-kT_s),\ Q(t) = 0$</td><td>0.5 pt<br>1 pt</td></tr>
<tr><td>A2. Sequence 1 1 0 1 → phases: 0°, 0°, 180°, 0° → $I(t)$ phase reversal at the 3rd symbol, $Q(t) = 0$<br><br></td><td>0.5 pt<br>0.5 pt</td></tr>
<tr><td>A3. 45° offset: $I' = A/\sqrt{2}$ → SNR degraded by 3 dB, detection still possible.<br>90° offset: $I' = 0$ → complete detection failure (all decisions are wrong).</td><td>0.5 pt<br>0.5 pt</td></tr>
<tr><td>A4. Corrected by a Costas loop / Use a DPSK encoding unsensitive to phase offset.</td><td>0.5 pt</td></tr>
<tr><td>A5. A residual offset $\Delta f$ causes the constellation to rotate continuously at rate $2\pi\Delta f$</td><td>0.5 pt</td></tr>
</table>

B] BFSK

1. Briefly explain the modulation principle.
2. Plot approximately the transmitted signal for the following binary sequence: **1 1 0 1**
3. Compare the PLL-based demodulator with the envelope detection-based demodulator.
4. Give the block diagram of each.

<table>
<tr><td>B1. bit=1 → frequency $f_1$, bit=0 → frequency $f_0$; information is encoded in the carrier frequency.</td><td>0.5 pt</td></tr>
<tr><td>B2. Sequence 1 1 0 1 → $f_1, f_1, f_0, f_1$ — visible frequency change at the 3rd symbol.<br><br></td><td>0.5 pt</td></tr>
<tr><td><table><tr><th></th><th>Coherent (PLL)</th><th>Non-coherent (envelope)</th></tr><tr><td>Phase sync</td><td>Required</td><td>Not required</td></tr><tr><td>Complexity</td><td>High</td><td>Low</td></tr><tr><td>Performance</td><td>Better (~3 dB gain)</td><td>Slightly worse</td></tr></table></td><td>0.5 pt<br>1 pt</td></tr>
<tr><td><br><br><br><br>4.</td><td>1 pt</td></tr>
</table>

**Reminders**

| | |
|---|---|
| $R_b = R_s \log_2 M$ | $C = B\log_2\left(1+\left(\frac{S}{N}\right)\right) = B\log_2\left(1+\frac{P}{N_0 B}\right)$ |
| $E_{bit} = \frac{E_{sym}}{log_2 M} = \frac{1}{M log_2 M}\sum_i E_{a_i}\, E_h$ | $S_a(f) = \sigma_a^2 + 2\sigma_a^2 \sum_{k=1} C_a(k)\cos(2\pi k f T_b) + \frac{\mu_a^2}{T_b}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T_b}\right)$ <br> $S_e(f) = \frac{1}{\boldsymbol{T_b}}\lvert H(f)\rvert^2\, S_a(f)$ |
| $g_r(t) = \frac{k\prime}{\sigma_B^2} h_m^*(t_0 - t) \qquad SNR_{t_0} = \frac{E_{h_m}}{\sigma_B^2}$ | $S_{opt} = \frac{a_0 + a_1}{2}E_h - \frac{N_0}{2(a_1 - a_0)}\, ln\frac{p_1}{p_0}$ |
| $\mathrm{Q}(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-\frac{t^2}{2}}dt$ | $P_e = Q\left(\frac{(a_1-a_0)E_h}{2\sigma_n}\right) = Q\left(\sqrt{\frac{(a_1-a_0)^2 E_h}{2N_0}}\right)$ |
| $RC_\alpha(t) = \frac{sin\left(\frac{\pi t}{T_s}\right)}{\frac{\pi t}{T_s}}\;\frac{cos\left(\frac{\alpha\pi t}{T_s}\right)}{1-\left(\frac{2\alpha t}{T_s}\right)^2}$ | $RC_\alpha(f) = \begin{cases} T_s & \text{si} \quad \lvert f\rvert \le \frac{1-\alpha}{2T_s} \\ T_s cos^2\left(\frac{\pi}{4\alpha}(2\lvert f\rvert T_s - (1-\alpha)\right) & \text{si} \quad \frac{1-\alpha}{2T_s} \le \lvert f\rvert \le \frac{1+\alpha}{2T_s} \\ 0 & ailleurs \end{cases}$ |
| $RRC_\alpha(t) = \frac{\sin\left(\pi\frac{t}{T_s}(1-\alpha)\right) + 4\alpha\frac{t}{T_s}\cos\left(\pi\frac{t}{T_s}(1+\alpha)\right)}{\pi\frac{t}{T_s}\left[1-\left(4\alpha\frac{t}{T_s}\right)^2\right]}$ | $RRC_\alpha(f) = \begin{cases} \sqrt{T_s} & \text{si} \quad \lvert f\rvert \le \frac{1-\alpha}{2T_s} \\ \sqrt{T_s} cos\left(\frac{\pi}{4\alpha}(2\lvert f\rvert T_s - (1-\alpha)\right) & \text{si} \quad \frac{1-\alpha}{2T_s} \le \lvert f\rvert \le \frac{1+\alpha}{2T_s} \\ 0 & ailleurs \end{cases}$ |
| $\eta = R_b/B = \frac{2\log_2 M}{1+\alpha}$ | $m(t) = \overbrace{\sum_k a_k h_e(t-kT_s)}^{\boldsymbol{i(t)}}\, A cos(2\pi f_c t) - \overbrace{\sum_k b_k h_e(t-kT_s)}^{\boldsymbol{q(t)}}\, A sin(2\pi f_c t)$ |
| $BER_{M-ASK} \approx \frac{2(M-1)}{M log_2 M} Q\left(\sqrt{\frac{6 log_2 M.E_b}{(M^2-1)N_0}}\right)$ | $BER_{BPSK+QPSK} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\; BER_{M-PSK} \approx \frac{2}{log_2 M} Q\left(\sqrt{\frac{2 log_2 M.E_b}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$ |
| $BER_{M-QAM} \approx \frac{4}{log_2 M}\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3 log_2 M.E_b}{(M-1)\,N_0}}\right)$ | $BER_{M-FSK(NC)} = \frac{1}{2} exp\left(-\frac{E_b}{2N_0}\right), BER_{M-FSK(C)} \approx \frac{(M-1)}{2} Q\left(\sqrt{\frac{log_2 M.E_b}{2N_0}}\right)$ |
| $y = \frac{A^2}{2}\cos(2\pi f_i \tau) \qquad \tau = \frac{1}{2(f_1 - f_0)} = \frac{1}{2\Delta f}$ | $\text{output} = arg(z[n]\cdot z^*[n-1]) = \varphi[n] - \varphi[n-1] \approx 2\pi f_{inst}\cdot T_s$ |