Received: 27 March 2024 | Revised: 5 July 2024 | Accepted: 14 August 2024

DOI: 10.1002/mana.202400150

MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

A criterion for the holomorphy of the curvature of smooth planar webs and applications to dual webs of homogeneous foliations on $\mathbb{P}^2_{\mathbb{C}}$

Samir Bedrouni[1] | **David Marín**[2,3]

[1]Faculté de Mathématiques, USTHB, BP 32, El-Alia, Bab-Ezzouar, Alger, Algeria

[2]Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, Barcelona, Spain

[3]Centre de Recerca Matemàtica, Edifici Cc, Campus de Bellaterra, Barcelona, Spain

Correspondence
Samir Bedrouni, Faculté de Mathématiques, USTHB, BP 32, El-Alia, 16111 Bab-Ezzouar, Alger, Algeria.
Email: sbedrouni@usthb.dz
David Marín, Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08193 Cerdanyola del Vallès, Barcelona, Spain.
Email: david.marin@uab.cat

Funding information
Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D, Grant/Award Number: CEX2020-001084-M; Ministry of Science, Innovation and Universities of Spain, Grant/Award Numbers: PGC2018-095998-B-I00, PID2021-125625NB-I00; Agency for Management of University and Research Grants of Catalonia, Grant/Award Numbers: 2017SGR1725, 2021SGR01015

Abstract

Let $d \geq 3$ be an integer. For a holomorphic d-web $\mathcal{W}$ on a complex surface M, smooth along an irreducible component D of its discriminant $\Delta(\mathcal{W})$, we establish an effective criterion for the holomorphy of the curvature of $\mathcal{W}$ along D, generalizing results on decomposable webs due to Marín, Pereira, and Pirio. As an application, we deduce a complete characterization for the holomorphy of the curvature of the Legendre transform (dual web) $\mathrm{Leg}\mathcal{H}$ of a homogeneous foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$, generalizing some of our previous results. This then allows us to study the flatness of the d-web $\mathrm{Leg}\mathcal{H}$ in the particular case where the foliation $\mathcal{H}$ is Galois. When the Galois group of $\mathcal{H}$ is cyclic, we show that $\mathrm{Leg}\mathcal{H}$ is flat if and only if $\mathcal{H}$ is given, up to linear conjugation, by one of the two 1-forms $\omega_1^d = y^d\mathrm{d}x - x^d\mathrm{d}y$, $\omega_2^d = x^d\mathrm{d}x - y^d\mathrm{d}y$. When the Galois group of $\mathcal{H}$ is noncyclic, we obtain that $\mathrm{Leg}\mathcal{H}$ is always flat.

KEYWORDS
curvature, Galois homogeneous foliation, Legendre transform, web

INTRODUCTION

A (regular) d-web $\mathcal{W}$ on $(\mathbb{C}^2, 0)$ is the data of a family $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_d\}$ of regular holomorphic foliations on $(\mathbb{C}^2, 0)$, which are pairwise transverse at the origin. We say that $\mathcal{W}$ is the superposition of the foliations $\mathcal{F}_1, \ldots, \mathcal{F}_d$ and we write $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_d$.

A (global) d-web on a complex surface M is given in a local chart (x, y) by an implicit differential equation $F(x, y, y') = 0$, where $F(x, y, p) = \sum_{i=0}^{d} a_i(x, y)p^{d-i}$ is a (reduced) polynomial in p of degree d, having analytic coefficients a_i, with a_0 not identically zero. In a neighborhood of every point $z_0 = (x_0, y_0)$, such that $a_0(x_0, y_0)\Delta(x_0, y_0) \neq 0$, where $\Delta(x, y)$ is the p-discriminant of F, the integral curves of this equation define a regular d-web on $(\mathbb{C}^2, z_0)$.

To every d-web $\mathcal{W}$ on M with $d \geq 3$, we can associate a meromorphic 2-form with poles along the discriminant $\Delta(\mathcal{W})$, called the curvature of $\mathcal{W}$ and denoted by $K(\mathcal{W})$, see Section 2.3. A web with zero curvature is called flat. When $M = \mathbb{P}^2_{\mathbb{C}}$, the flatness of a web $\mathcal{W}$ on $\mathbb{P}^2_{\mathbb{C}}$ is characterized by the holomorphy of its curvature $K(\mathcal{W})$ along the generic points of $\Delta(\mathcal{W})$.

In 2008, Pereira and Pirio [8, Theorem 7.1] established a result on the holomorphy of the curvature of a completely decomposable d-web $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_d$. In 2013, Marín and Pereira [7, Theorem 1] extended this result to decomposable webs of the form $\mathcal{W} = \mathcal{W}_2 \boxtimes \mathcal{W}_{d-2}$, that is, which are the superposition of the (local) foliations of a 2-web $\mathcal{W}_2$ and a $(d-2)$-web $\mathcal{W}_{d-2}$. In this paper, we establish an effective criterion (Theorem 2.1) for the holomorphy of the curvature of a d-web $\mathcal{W}$ defined on a complex surface and smooth along an irreducible component of its discriminant $\Delta(\mathcal{W})$, generalizing these two results (see Corollary 2.6 and Remark 2.7).

We are then interested in the foliations on $\mathbb{P}^2_{\mathbb{C}}$, which are *homogeneous*, that is, which are invariant by homotheties. In [3, section 3] we studied, for a homogeneous foliation $\mathcal{H}$ of degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$, the problem of the flatness of its Legendre transform (its dual web) $\mathrm{Leg}\mathcal{H}$; it is a d-web on the dual projective plane $\check{\mathbb{P}}^2_{\mathbb{C}}$ whose leaves are the tangent lines to the leaves of $\mathcal{H}$, see Section 4. Theorem 2.1 allows us to establish, for such a foliation $\mathcal{H}$, a complete characterization (Theorem 3.1) of the holomorphy of the curvature of the d-web $\mathrm{Leg}\mathcal{H}$ along an irreducible component of the discriminant $\Delta(\mathrm{Leg}\mathcal{H})$, generalizing our results in [3, Theorems 3.5 and 3.8] (see Corollary 3.6 and Remark 3.7).

We finally focus on the particular case of a homogeneous foliation $\mathcal{H}$ of degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$, which is Galois in the sense of [2, Definition 6.16], see Section 5. When the Galois group of $\mathcal{H}$ is cyclic, we prove that $\mathrm{Leg}\mathcal{H}$ is flat if and only if, up to linear conjugation, $\mathcal{H}$ is given by one of the two 1-forms $\omega_1^d = y^d\mathrm{d}x - x^d\mathrm{d}y$, $\omega_2^d = x^d\mathrm{d}x - y^d\mathrm{d}y$. When the Galois group of $\mathcal{H}$ is noncyclic, we show that $\mathrm{Leg}\mathcal{H}$ is always flat, see Theorem 4.4.

1 | PRELIMINARIES

1.1 | Webs

Let $d \geq 1$ be a integer. A *(global) d-web* $\mathcal{W}$ on a complex surface M is given by an open covering $(U_i)_{i\in I}$ of M and a collection of d-symmetric 1-forms $\omega_i \in \mathrm{Sym}^d\Omega^1_M(U_i)$, with isolated zeros, satisfying:

($\mathfrak{a}$) there exists $g_{ij} \in \mathcal{O}^*_M(U_i \cap U_j)$ such that ω_i coincides with $g_{ij}\omega_j$ on $U_i \cap U_j$;
($\mathfrak{b}$) for every generic point m of U_i, $\omega_i(m)$ factors as the product of d pairwise linearly independent 1-forms.

The *discriminant* $\Delta(\mathcal{W})$ of $\mathcal{W}$ is the divisor on M defined locally by $\Delta(\omega_i) = 0$, where $\Delta(\omega_i)$ is the discriminant of the d-symmetric 1-form $\omega_i \in \mathrm{Sym}^d\Omega^1_M(U_i)$, see [9, Chapter 1, section 1.3.4]. The support of $\Delta(\mathcal{W})$ consists of the points of M, which do not satisfy condition ($\mathfrak{b}$). When $d = 1$, this condition is always satisfied and we recover the usual definition of a holomorphic foliation $\mathcal{F}$ on M.

The *tangent locus* $\mathrm{T}_m\mathcal{W}$ of $\mathcal{W}$ at a point $m \in U_i \setminus \Delta(\mathcal{W})$ is the union of the d kernels at m of the linear factors of $\omega_i(m)$.

A global d-web $\mathcal{W}$ on M is said to be *decomposable* if there are global webs $\mathcal{W}_1, \mathcal{W}_2$ on M sharing no common subwebs such that $\mathcal{W}$ is the superposition of $\mathcal{W}_1$ and $\mathcal{W}_2$; we then write $\mathcal{W} = \mathcal{W}_1 \boxtimes \mathcal{W}_2$. Otherwise $\mathcal{W}$ is said to be *irreducible*. We say that $\mathcal{W}$ is *completely decomposable* if there exist global foliations $\mathcal{F}_1, \dots, \mathcal{F}_d$ on M such that $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_d$. For more details on this subject, we refer to [9].

1.2 | Characteristic surface of a web

Let $\mathcal{W}$ be a holomorphic d-web on a complex surface M. Let $\widetilde{M} = \mathbb{P}\mathrm{T}^*M$ be the projectivization of the cotangent bundle of M; the *characteristic surface* of $\mathcal{W}$ is the surface $S_{\mathcal{W}} \subset \widetilde{M}$ defined by

$$S_{\mathcal{W}} := \overline{\left\{(m, [\eta]) \in \widetilde{M} \middle| m \in M \setminus \Delta(\mathcal{W}), \ker\eta \subset \mathrm{T}_m\mathcal{W}\right\}}.$$

We will give a local expression of this surface. First of all, let us consider a local coordinate system (x, y) on an open subset U of M. Denote by $\pi : \widetilde{M} \to M$ the natural projection. We define a coordinate system on the open set $\pi^{-1}(U)$, by denoting by $(x, y, [p : q])$ the coordinates of the point $(m, [q\mathrm{d}y - p\mathrm{d}x]) \in \pi^{-1}(U)$, where (x, y) are the local coordinates of m in U. If $\mathcal{W}$ is given on U by the d-symmetric 1-form $\omega = \sum_{i=0}^{d} a_i(x, y)(\mathrm{d}x)^i(\mathrm{d}y)^{d-i}$, with $a_i \in \mathcal{O}_M(U)$, then

$$S_{\mathcal{W}} \cap \pi^{-1}(U) = \{(x, y, [p : q]) \in \widetilde{M} | \widetilde{F}(x, y, p, q) = 0\},$$

where $\widetilde{F}(x, y, p, q) = \sum_{i=0}^{d} a_i(x, y)p^{d-i}q^i$.

In the sequel, we will work in the affine chart $(U_q, (x, y, p))$ defined by $U_q := \pi^{-1}(U) \setminus \{q = 0\}$ and $p := [p : 1]$. Setting $F(x, y, p) := \widetilde{F}(x, y, p, 1) = \sum_{i=0}^{d} a_i(x, y)p^{d-i}$, we have

$$S_{\mathcal{W}} \cap U_q = \{(x, y, p) \in \widetilde{M} | F(x, y, p) = 0\}.$$

We will denote by $\pi_{\mathcal{W}} : S_{\mathcal{W}} \to M$ the restriction of π to $S_{\mathcal{W}}$. Let us introduce the following definition, which will be useful later.

Definition 1.1. With the above notations, let D be an irreducible component of the discriminant $\Delta(\mathcal{W})$. We will say that $\mathcal{W}$ is *smooth along* D if for every generic point m of D, the characteristic surface $S_{\mathcal{W}}$ of $\mathcal{W}$ is smooth at every point of the fiber $\pi_{\mathcal{W}}^{-1}(m)$.

Example 1.2. On $M = \mathbb{C}^2$, the 2-web $\mathcal{W}$ given by $\omega = (y^2 - x)\mathrm{d}y^2 + 2x\mathrm{d}x\mathrm{d}y - x\mathrm{d}x^2$ has discriminant $\Delta(\mathcal{W}) = 4xy^2$ and its characteristic surface $S_{\mathcal{W}}$ has equation $F(x, y, p) := (y^2 - x)p^2 + 2xp - x = 0$. Note that $\mathcal{W}$ is smooth along the irreducible component $D_1 := \{x = 0\} \subset \Delta(\mathcal{W})$. Indeed, the fiber $\pi_{\mathcal{W}}^{-1}(m)$ over a generic point $m = (0, y) \in D_1$ is reduced to the point $\widetilde{m} = (0, y, 0)$, and the surface $S_{\mathcal{W}}$ is smooth at $\widetilde{m}$, because $\partial_x F(0, y, 0) = -1 \neq 0$. However, $\mathcal{W}$ is not smooth along the irreducible component $D_2 := \{y = 0\} \subset \Delta(\mathcal{W})$, because, for every generic point $m = (x, 0) \in D_2$, we have $\pi_{\mathcal{W}}^{-1}(m) = \{(x, 0, 1)\}$ and $\partial_x F(x, 0, 1) \equiv \partial_y F(x, 0, 1) \equiv \partial_p F(x, 0, 1) \equiv 0$.

1.3 | Fundamental form, curvature, and flatness of a web

We recall here the definitions of the fundamental form and the curvature of a d-web $\mathcal{W}$. Let us first suppose that $\mathcal{W}$ is a germ of completely decomposable d-web on $(\mathbb{C}^2, 0)$, $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_d$. For each $1 \leq i \leq d$, let ω_i be a 1-form with at most an isolated singularity at 0 defining the foliation $\mathcal{F}_i$. According to [8], for every triple (r, s, t) with $1 \leq r < s < t \leq d$, we define $\eta_{rst} = \eta(\mathcal{F}_r \boxtimes \mathcal{F}_s \boxtimes \mathcal{F}_t)$ as the unique meromorphic 1-form satisfying the following equalities:

$$\begin{cases} \mathrm{d}(\delta_{st}\,\omega_r) = \eta_{rst} \wedge \delta_{st}\,\omega_r \\ \mathrm{d}(\delta_{tr}\,\omega_s) = \eta_{rst} \wedge \delta_{tr}\,\omega_s \\ \mathrm{d}(\delta_{rs}\,\omega_t) = \eta_{rst} \wedge \delta_{rs}\,\omega_t, \end{cases} \tag{1.1}$$

where δ_{ij} denotes the function defined by the relation $\omega_i \wedge \omega_j = \delta_{ij}\,\mathrm{d}x \wedge \mathrm{d}y$. We call *fundamental form* of the web $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_d$ the 1-form

$$\eta(\mathcal{W}) = \eta(\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_d) = \sum_{1 \leq r < s < t \leq d} \eta_{rst}. \tag{1.2}$$

We can easily verify that $\eta(\mathcal{W})$ is a meromorphic 1-form with poles along the discriminant $\Delta(\mathcal{W})$ of $\mathcal{W}$, and that it is well defined up to addition of a closed logarithmic 1-form $\dfrac{\mathrm{d}g}{g}$ with $g \in \mathcal{O}^*(\mathbb{C}^2, 0)$ (cf. [3, 10]).

Now, if $\mathcal{W}$ is an arbitrary d-web on a complex surface M, then we can transform it into a completely decomposable d-web by taking its pull-back by a suitable ramified Galois covering. The invariance of the fundamental form of this new web by the action of the Galois group allows us to descend it to a global meromorphic 1-form $\eta(\mathcal{W})$ on M, with poles along the discriminant of $\mathcal{W}$ (see [7]).

The *curvature* of the web $\mathcal{W}$ is by definition the 2-form

$$K(\mathcal{W}) = \mathrm{d}\,\eta(\mathcal{W}).$$

It is a meromorphic 2-form with poles along the discriminant $\Delta(\mathcal{W})$, canonically associated to $\mathcal{W}$; more precisely, for any dominant holomorphic map φ, we have $K(\varphi^*\mathcal{W}) = \varphi^*K(\mathcal{W})$.

A d-web $\mathcal{W}$ is called *flat* if its curvature $K(\mathcal{W})$ vanishes identically.

Note that a d-web $\mathcal{W}$ on $\mathbb{P}^2_{\mathbb{C}}$ is flat if and only if its curvature is holomorphic over the generic points of the irreducible components of $\Delta(\mathcal{W})$. This follows from the definition of $K(\mathcal{W})$ and the fact that there are no holomorphic 2-forms on $\mathbb{P}^2_{\mathbb{C}}$ other than the zero 2-form.

2 | CRITERION FOR THE HOLOMORPHY OF THE CURVATURE OF SMOOTH WEBS

In this section, we propose to establish the following theorem.

Theorem 2.1. *Let $\mathcal{W}$ be a holomorphic d-web on a complex surface M and let D be an irreducible component of the discriminant $\Delta(\mathcal{W})$. Assume that $\mathcal{W}$ is smooth along D. Then, the fundamental form $\eta(\mathcal{W})$ has simple poles along D. More precisely, choose a local coordinate system (x, y) on M such that $D = \{y = 0\}$ and let $F(x, y, p) = 0$, $p = \frac{\mathrm{d}y}{\mathrm{d}x}$, be an implicit differential equation defining $\mathcal{W}$. Write $F(x, 0, p) = a_0(x)\prod_{\alpha=1}^{n}(p - \varphi_\alpha(x))^{\nu_\alpha}$ with $\varphi_\alpha \not\equiv \varphi_\beta$ if $\alpha \neq \beta$. Then, the 1-form*

$$\eta(\mathcal{W}) - \frac{1}{6y}\sum_{\alpha=1}^{n}(\nu_\alpha - 1)(\psi_\alpha(x)(\mathrm{d}y - \varphi_\alpha(x)\mathrm{d}x) + (\nu_\alpha - 2)\mathrm{d}y)$$

is holomorphic along $D = \{y = 0\}$, where ψ_α is a function of the coordinate x defined, for all $\alpha \in \{1, \dots, n\}$ such that $\nu_\alpha \geq 2$, by

$$\psi_\alpha(x) = \frac{1}{\nu_\alpha}\left[(\nu_\alpha - 2)\left(d - \varphi_\alpha(x)\frac{\partial_p\partial_y F(x, 0, \varphi_\alpha(x))}{\partial_y F(x, 0, \varphi_\alpha(x))}\right) - 2(\nu_\alpha + 1)\sum_{\beta=1,\beta\neq\alpha}^{n}\frac{\nu_\beta\varphi_\beta(x)}{\varphi_\alpha(x) - \varphi_\beta(x)}\right].$$

In particular, the curvature $K(\mathcal{W})$ is holomorphic along D if and only if

$$\sum_{\alpha=1}^{n}(\nu_\alpha - 1)\varphi_\alpha(x)\psi_\alpha(x) \equiv 0 \qquad \text{and} \qquad \sum_{\alpha=1}^{n}(\nu_\alpha - 1)\frac{\mathrm{d}}{\mathrm{d}x}\psi_\alpha(x) \equiv 0.$$

Remark 2.2. When the component $D \subset \Delta(\mathcal{W})$ is totally invariant by $\mathcal{W}$, the curvature $K(\mathcal{W})$ is always holomorphic along D.

Remark 2.3. Assume that $\nu_\alpha = \nu \geq 2$ for all $\alpha \in \{1, \dots, n\}$. The following assertions hold:

1. If $\nu = 2$ (which implies that d is even), then the curvature $K(\mathcal{W})$ is always holomorphic along D.
2. If $\nu \geq 3$, then the curvature $K(\mathcal{W})$ is holomorphic along D if and only if

$$\sum_{\alpha=1}^{n}\varphi_\alpha(x)(d - \rho_\alpha(x)) \equiv 0 \qquad \text{and} \qquad \sum_{\alpha=1}^{n}\frac{\mathrm{d}}{\mathrm{d}x}\rho_\alpha(x) \equiv 0,$$

where $\rho_\alpha(x) := \varphi_\alpha(x)\dfrac{\partial_p\partial_y F(x, 0, \varphi_\alpha(x))}{\partial_y F(x, 0, \varphi_\alpha(x))}$.

Indeed, it suffices to set $f_{\alpha,\beta} = \frac{\varphi_\beta}{\varphi_\alpha - \varphi_\beta}$ and to note that

$$\sum_{\alpha=1}^{n} \sum_{\beta=1,\beta\neq\alpha}^{n} \varphi_\alpha f_{\alpha,\beta} = \sum_{1\leq\alpha<\beta\leq n} \left(\varphi_\alpha f_{\alpha,\beta} + \varphi_\beta f_{\beta,\alpha}\right) \equiv 0$$

and

$$\sum_{\alpha=1}^{n} \sum_{\beta=1,\beta\neq\alpha}^{n} f_{\alpha,\beta} = \sum_{1\leq\alpha<\beta\leq n} \left(f_{\alpha,\beta} + f_{\beta,\alpha}\right) \equiv -\binom{n}{2} \equiv \text{constant.}$$

The hypothesis of smoothness of $\mathcal{W}$ along the component $D \subset \Delta(\mathcal{W})$ is essential for the validity of Theorem 3.1, as the following example shows.

Example 2.4. Let M be a complex surface and let $\mathcal{W}$ be the 3-web defined in local coordinates (x, y) by the differential equation

$$F(x,y,p) := \big(\lambda(x^2-1)p + (x-3)y^\kappa\big)\big(\lambda(x^2-1)p + (x+3)y^\kappa\big)\big(\lambda(x^2-1)p - 2xy^\kappa\big) = 0,$$

where $p = \frac{\mathrm{d}y}{\mathrm{d}x}$, $\kappa \in \mathbb{N} \setminus \{0,1\}$, $\lambda \in \mathbb{C}^*$. For this web, we have

$$\Delta(\mathcal{W}) = 2916\lambda^6(x^2-1)^8 y^{6\kappa} \qquad \text{and} \qquad \eta(\mathcal{W}) = \frac{7\mathrm{d}(x^2-1)}{3(x^2-1)} + \left(\frac{2\kappa}{y} - \frac{\lambda}{3y^\kappa}\right)\mathrm{d}y.$$

We see that $\eta(\mathcal{W})$ is closed and therefore that $\mathcal{W}$ is flat. Moreover, $\eta(\mathcal{W})$ has poles of order $\kappa > 1$ along the component $D := \{y = 0\} \subset \Delta(\mathcal{W})$. Note that $\mathcal{W}$ is not smooth along D. Indeed, the fiber $\pi_{\mathcal{W}}^{-1}(m)$ over a generic point $m = (x, 0) \in D$ consists of the single point $\widetilde{m} = (x, 0, 0)$ and the surface $S_{\mathcal{W}}$ is not smooth at $\widetilde{m}$, because $\partial_x F(x,0,0) \equiv \partial_y F(x,0,0) \equiv \partial_p F(x,0,0) \equiv 0$.

Remark 2.5. In [5, p. 286], the author claimed that the fundamental form of a planar 3-web $\mathcal{W}$ has probably at most simple poles along $\Delta(\mathcal{W})$ and he gave an argument in the particular case where $\mathcal{W}$ is defined by a differential equation of type $a_0(x,y)p^3 + a_2(x,y)p + a_3(x,y) = 0$, $p = \frac{\mathrm{d}y}{\mathrm{d}x}$. Note that the 3-web given in Example 2.4 is of this type and its fundamental form has no simple poles along $y = 0$ if $\kappa > 1$. This contradicts the claim of [5, p. 286].

Corollary 2.6. *Let $\mathcal{W}$ be a holomorphic d-web on a complex surface M and let D be an irreducible component of the discriminant $\Delta(\mathcal{W})$. Assume that $\mathcal{W}$ is smooth along D. Fix a local coordinate system (x, y) on M such that $D = \{y = 0\}$ and let $F(x,y,p) = 0$, $p = \frac{\mathrm{d}y}{\mathrm{d}x}$, be an implicit differential equation defining $\mathcal{W}$. Assume moreover that $F(x,0,p) = a_0(x)(p-\varphi_0(x))^\nu \prod_{\alpha=1}^{d-\nu}(p-\varphi_\alpha(x))$ with $\varphi_\alpha \neq \varphi_0$ for all $\alpha \in \{1,\dots,d-\nu\}$ and $\varphi_\alpha \not\equiv \varphi_\beta$ if $\alpha \neq \beta$. Then, the curvature $K(\mathcal{W})$ is holomorphic on D if and only if $\varphi_0 \equiv 0$ or $\psi \equiv 0$, where*

$$\psi(x) = (\nu-2)\left(d - \varphi_0(x)\frac{\partial_p\partial_y F(x,0,\varphi_0(x))}{\partial_y F(x,0,\varphi_0(x))}\right) - 2(\nu+1)\sum_{\alpha=1}^{d-\nu}\frac{\varphi_\alpha(x)}{\varphi_0(x)-\varphi_\alpha(x)}.$$

Remark 2.7. In a neighborhood of every generic point of D, the d-web $\mathcal{W}$ decomposes as $\mathcal{W} = \mathcal{W}_\nu \boxtimes \mathcal{W}_{d-\nu}$ with

$$\mathcal{W}_\nu\big|_D : \mathrm{d}y - \varphi_0(x)\mathrm{d}x = 0 \qquad \text{and} \qquad \mathcal{W}_{d-\nu}\big|_D : \prod_{\alpha=1}^{d-\nu}(\mathrm{d}y - \varphi_\alpha(x)\mathrm{d}x) = 0.$$

When $\nu = 2$, we recover the barycenter criterion, namely, Theorem 1 of [7] (see also [8, Theorem 7.1]): The curvature of $\mathcal{W} = \mathcal{W}_2 \boxtimes \mathcal{W}_{d-2}$ is holomorphic on D if and only if D is invariant by $\mathcal{W}_2$ or by the barycenter $\beta_{\mathcal{W}_2}(\mathcal{W}_{d-2})$ of

$\mathcal{W}_{d-2}$ with respect to $\mathcal{W}_2$. Indeed, on the one hand, the invariance of $D = \{y = 0\}$ by $\mathcal{W}_2$ translates into $\varphi_0 \equiv 0$. On the other hand, the restriction of $\beta_{\mathcal{W}_2}(\mathcal{W}_{d-2})$ to D is given by

$$\mathrm{d}y - \left[\varphi_0(x) + \frac{1}{\frac{1}{d-2}\sum_{\alpha=1}^{d-2}\frac{1}{\varphi_\alpha(x)-\varphi_0(x)}}\right]\mathrm{d}x,$$

or equivalently, by

$$\sum_{\alpha=1}^{d-2}\frac{1}{\varphi_0(x)-\varphi_\alpha(x)}\mathrm{d}y + \left[d-2-\varphi_0(x)\sum_{\alpha=1}^{d-2}\frac{1}{\varphi_0(x)-\varphi_\alpha(x)}\right]\mathrm{d}x = \sum_{\alpha=1}^{d-2}\frac{1}{\varphi_0(x)-\varphi_\alpha(x)}\mathrm{d}y - \sum_{\alpha=1}^{d-2}\frac{\varphi_\alpha(x)}{\varphi_0(x)-\varphi_\alpha(x)}\mathrm{d}x,$$

so that the invariance of D by $\beta_{\mathcal{W}_2}(\mathcal{W}_{d-2})$ is characterized by $\sum_{\alpha=1}^{d-2}\frac{\varphi_\alpha}{\varphi_0-\varphi_\alpha} \equiv 0$ and therefore by $\psi \equiv 0$, because $\psi = -6\sum_{\alpha=1}^{d-2}\frac{\varphi_\alpha}{\varphi_0-\varphi_\alpha}$.

The proof of Theorem 3.1 consists essentially in determining the principal part of the Laurent series of the fundamental form $\eta(\mathcal{W})$ along the component $D = \{y = 0\}$ of the discriminant of $\mathcal{W}$. To do this, we need the following lemma.

Lemma 2.8. *The fundamental form of the 3-web $\mathcal{W}$ defined by the 1-forms $\omega_\ell = \mathrm{d}y - \lambda_\ell(x,y)\mathrm{d}x$, $\ell = 1,2,3$, is given by*

$$\eta(\mathcal{W}) = \sum_{(i,j,k)\in\langle 1,2,3\rangle}\frac{\partial_y(\lambda_i\lambda_j)-\partial_x\lambda_k}{(\lambda_i-\lambda_k)(\lambda_j-\lambda_k)}(\mathrm{d}y-\lambda_k\mathrm{d}x),$$

where $\langle 1,2,3\rangle := \{(1,2,3),(3,1,2),(2,3,1)\}$.

Proof. This follows from a straightforward computation using formula (1.1). □

Proof of Theorem 2.1. In a neighborhood of every generic point m of D, the web $\mathcal{W}$ decomposes as $\mathcal{W} = \boxtimes_{\alpha=1}^n\mathcal{W}_\alpha$, where $\mathcal{W}_\alpha$ is a ν_α-web having a unique slope $p = \varphi_\alpha(x)$ along $y = 0$, that is, $\mathcal{W}_\alpha = \boxtimes_{i=1}^{\nu_\alpha}\mathcal{F}_i^\alpha$ and $\mathcal{F}_i^\alpha|_{y=0} : \mathrm{d}y - \varphi_\alpha(x)\mathrm{d}x = 0$. Then, $\eta(\mathcal{W}) = \eta_1 + \eta_2 + \eta_3$, where

$$\eta_1 = \sum_{\substack{\alpha=1\\ \nu_\alpha\geq 3}}^n\sum_{1\leq i<j<k\leq\nu_\alpha}\eta_{ijk}^{\alpha\alpha\alpha},\quad \eta_2 = \sum_{\substack{\alpha=1\\ \nu_\alpha\geq 2}}^n\sum_{1\leq i<j\leq\nu_\alpha}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^n\sum_{k=1}^{\nu_\beta}\eta_{ijk}^{\alpha\alpha\beta},\quad \eta_3 = \sum_{1\leq\alpha<\beta<\gamma\leq n}\sum_{\substack{1\leq i\leq\nu_\alpha\\ 1\leq j\leq\nu_\beta\\ 1\leq k\leq\nu_\gamma}}\eta_{ijk}^{\alpha\beta\gamma},$$

and $\eta_{ijk}^{\alpha\alpha\alpha}$, resp. $\eta_{ijk}^{\alpha\alpha\beta}$, resp. $\eta_{ijk}^{\alpha\beta\gamma}$, is the fundamental form of the 3-subweb $\mathcal{F}_i^\alpha\boxtimes\mathcal{F}_j^\alpha\boxtimes\mathcal{F}_k^\alpha$, resp. $\mathcal{F}_i^\alpha\boxtimes\mathcal{F}_j^\alpha\boxtimes\mathcal{F}_k^\beta$, resp. $\mathcal{F}_i^\alpha\boxtimes\mathcal{F}_j^\beta\boxtimes\mathcal{F}_k^\gamma$, of $\mathcal{W}$.

If $\alpha<\beta<\gamma$, then $(\varphi_\alpha-\varphi_\beta)(\varphi_\beta-\varphi_\gamma)(\varphi_\gamma-\varphi_\alpha)\not\equiv 0$, which implies, thanks to Lemma 2.8, that the 1-form $\eta_{ijk}^{\alpha\beta\gamma}$ has no poles along $y = 0$; therefore, the same is true for the 1-form η_3.

As for η_1 and η_2, let us first fix $\alpha\in\{1,\ldots,n\}$ such that $\nu_\alpha\geq 2$. Then, $\partial_x F(x,0,\varphi_\alpha(x))\equiv\partial_p F(x,0,\varphi_\alpha(x))\equiv 0$; the hypothesis of smoothness of $\mathcal{W}$ along $D = \{y=0\}$ implies that $\partial_y F(x,0,\varphi_\alpha(x))\not\equiv 0$. Put $z = p-\varphi_\alpha(x)$ and $F_\alpha(x,y,z) := F(x,y,z+\varphi_\alpha(x)) = \sum_{k\geq 0}F_{\alpha,k}(x,z)y^k$ with $F_{\alpha,k}\in\mathbb{C}\{x\}[z]$. Since $F_{\alpha,1}(x,0) = \partial_y F(x,0,\varphi_\alpha(x))\not\equiv 0$, the series $\Psi(y) := \sum_{k\geq 1}F_{\alpha,k}y^k$ is invertible and its inverse writes as $\Psi^{-1}(w) = \frac{1}{F_{\alpha,1}}w - \frac{F_{\alpha,2}}{(F_{\alpha,1})^3}w^2 + \cdots$. Moreover, define $U_\alpha\in\mathbb{C}\{x\}[z]$ by $F_\alpha(x,0,z) = z^{\nu_\alpha}U_\alpha(x,z)$; note that

$$U_\alpha(x,z) = a_0(x)\prod_{\beta=1,\beta\neq\alpha}^n\left(z+\varphi_\alpha(x)-\varphi_\beta(x)\right)^{\nu_\beta} = \sum_{k=0}^{d-\nu_\alpha}U_{\alpha,k}(x)z^k,$$

with $\frac{U_{\alpha,1}(x)}{U_{\alpha,0}(x)} = \frac{\partial_z U_\alpha(x,0)}{U_\alpha(x,0)} = \sum_{\beta=1,\beta\neq\alpha}^{n} \frac{\nu_\beta}{\varphi_\alpha(x)-\varphi_\beta(x)}$. Writing $F_{\alpha,1}(x,z) = \sum_{k=0}^{d} G_{\alpha,k}(x) z^k$, with $G_{\alpha,0} \not\equiv 0$, it follows that in a neighborhood of $(x,0,0)$, the equation $F_\alpha(x,y,z) = 0$ is equivalent to

$$y = (\Psi^{-1}(-F_{\alpha,0}))(x,z) = -z^{\nu_\alpha} \frac{U_\alpha(x,z)}{F_{\alpha,1}(x,z)} - z^{2\nu_\alpha} \frac{F_{\alpha,2}(x,z)(U_\alpha(x,z))^2}{(F_{\alpha,1}(x,z))^3} + \cdots = Y_{\alpha,0}(x) z^{\nu_\alpha} + Y_{\alpha,1}(x) z^{\nu_\alpha+1} + \cdots =: Y_\alpha(x,z),$$

with $Y_{\alpha,0} = -\frac{U_{\alpha,0}}{G_{\alpha,0}} \not\equiv 0$ and $Y_{\alpha,1} = \frac{G_{\alpha,1}U_{\alpha,0} - G_{\alpha,0}U_{\alpha,1}}{(G_{\alpha,0})^2}$ because $\nu_\alpha \geq 2$. Thus, we can write $Y_\alpha(x,z) = (X_\alpha(x,z))^{\nu_\alpha}$ with $X_\alpha(x,z) = \sum_{k\geq 1} X_{\alpha,k}(x) z^k, X_{\alpha,1} = \left(Y_{\alpha,0}\right)^{\frac{1}{\nu_\alpha}} \not\equiv 0$ and $\frac{X_{\alpha,2}}{X_{\alpha,1}} = \frac{Y_{\alpha,1}}{\nu_\alpha Y_{\alpha,0}}$. Then, the series $\Phi(z) := \sum_{k\geq 1} X_{\alpha,k} z^k$ is invertible and its inverse is of the form $\Phi^{-1}(w) = \sum_{k\geq 1} f_{\alpha,k} w^k$ with $f_{\alpha,k} \in \mathbb{C}\{x\}$, $f_{\alpha,1} = \frac{1}{X_{\alpha,1}}$ and $f_{\alpha,2} = -\frac{X_{\alpha,2}}{(X_{\alpha,1})^3}$. Therefore, the equality $y = (\Psi^{-1}(-F_{\alpha,0}))(x,z)$ is equivalent to $z = (\Phi^{-1}(y^{\frac{1}{\nu_\alpha}}))(x)$ and therefore to $p = (\Phi^{-1}(y^{\frac{1}{\nu_\alpha}}))(x) + \varphi_\alpha(x)$. As a result, in a neighborhood of m, the slopes p_j $(j = 1, \dots, \nu_\alpha)$ of $\mathrm{T}_{(x,y)}\mathcal{W}_\alpha$ are given by

$$p_j = \lambda_{\alpha,j}(x,y) := \varphi_\alpha(x) + \sum_{k\geq 1} f_{\alpha,k}(x) \zeta_\alpha^{jk} y^{\frac{k}{\nu_\alpha}}, \quad \text{where} \zeta_\alpha = \exp(\frac{2i\pi}{\nu_\alpha}).$$

Note furthermore that

$$\begin{aligned}\frac{f_{\alpha,2}}{(f_{\alpha,1})^2} &= -\frac{X_{\alpha,2}}{X_{\alpha,1}} = -\frac{Y_{\alpha,1}}{\nu_\alpha Y_{\alpha,0}} = \frac{1}{\nu_\alpha}\left(\frac{G_{\alpha,1}}{G_{\alpha,0}} - \frac{U_{\alpha,1}}{U_{\alpha,0}}\right) = \frac{1}{\nu_\alpha}\left[\left(\frac{\partial_z F_{\alpha,1}}{F_{\alpha,1}}\right)\Big|_{z=0} - \frac{U_{\alpha,1}}{U_{\alpha,0}}\right] \\ &= \frac{1}{\nu_\alpha}\left[\left(\frac{\partial_z\partial_y F_\alpha}{\partial_y F_\alpha}\right)\Big|_{(y,z)=(0,0)} - \sum_{\beta=1,\beta\neq\alpha}^{n} \frac{\nu_\beta}{\varphi_\alpha - \varphi_\beta}\right].\end{aligned} \tag{3.1}$$

We will now apply Lemma 2.8 to compute $\eta_{ijk}^{\alpha\alpha\alpha}$. Setting $w_\alpha = y^{\frac{1}{\nu_\alpha}}$, we obtain

$$\begin{aligned}&\partial_x \lambda_{\alpha,k} = \varphi_\alpha' + f_{\alpha,1}' \zeta_\alpha^k w_\alpha + f_{\alpha,2}' \zeta_\alpha^{2k} w_\alpha^2 + f_{\alpha,3}' \zeta_\alpha^{3k} w_\alpha^3 + \cdots, \\ &\partial_y(\lambda_{\alpha,i}\lambda_{\alpha,j}) = \frac{w_\alpha}{\nu_\alpha y}\Big[\varphi_\alpha f_{\alpha,1}(\zeta_\alpha^i + \zeta_\alpha^j) + 2\Big(\varphi_\alpha f_{\alpha,2}(\zeta_\alpha^{2i} + \zeta_\alpha^{2j}) + f_{\alpha,1}^2 \zeta_\alpha^{i+j}\Big) w_\alpha \\ &+ 3\Big(\varphi_\alpha f_{\alpha,3}(\zeta_\alpha^{3i} + \zeta_\alpha^{3j}) + f_{\alpha,1} f_{\alpha,2}(\zeta_\alpha^{2i+j} + \zeta_\alpha^{i+2j})\Big) w_\alpha^2 + \cdots\Big], \\ &(\lambda_{\alpha,i} - \lambda_{\alpha,k})(\lambda_{\alpha,j} - \lambda_{\alpha,k}) = w_\alpha^2(\zeta_\alpha^i - \zeta_\alpha^k)(\zeta_\alpha^j - \zeta_\alpha^k)\Big[f_{\alpha,1}^2 + f_{\alpha,1} f_{\alpha,2}(\zeta_\alpha^i + \zeta_\alpha^j + 2\zeta_\alpha^k) w_\alpha + \cdots\Big].\end{aligned}$$

According to Lemma 2.8, we have $\eta_{ijk}^{\alpha\alpha\alpha} = a_{ijk}(x,y)\mathrm{d}x + b_{ijk}(x,y)\mathrm{d}y$, where

$$\begin{aligned}a_{ijk} &= -\frac{\left(\partial_y(\lambda_{\alpha,i}\lambda_{\alpha,j}) - \partial_x\lambda_{\alpha,k}\right)\lambda_{\alpha,k}}{(\lambda_{\alpha,i} - \lambda_{\alpha,k})(\lambda_{\alpha,j} - \lambda_{\alpha,k})} - \frac{\left(\partial_y(\lambda_{\alpha,k}\lambda_{\alpha,j}) - \partial_x\lambda_{\alpha,i}\right)\lambda_{\alpha,i}}{(\lambda_{\alpha,k} - \lambda_{\alpha,i})(\lambda_{\alpha,j} - \lambda_{\alpha,i})} - \frac{\left(\partial_y(\lambda_{\alpha,i}\lambda_{\alpha,k}) - \partial_x\lambda_{\alpha,j}\right)\lambda_{\alpha,j}}{(\lambda_{\alpha,i} - \lambda_{\alpha,j})(\lambda_{\alpha,k} - \lambda_{\alpha,j})} \\ &= -\frac{1}{\nu_\alpha y}\left[\frac{\varphi_\alpha}{f_{\alpha,1}^2}\left(f_{\alpha,1}^2 - \varphi_\alpha f_{\alpha,2}\right) + 2\frac{\varphi_\alpha^2}{f_{\alpha,1}^3}\left(\zeta_\alpha^i + \zeta_\alpha^j + \zeta_\alpha^k\right)\left(f_{\alpha,2}^2 - f_{\alpha,1}f_{\alpha,3}\right) w_\alpha + A_{-1} w_\alpha^2\right] \\ &+ A_0, \quad \text{with} \quad A_{-1}, A_0 \in \mathbb{C}\{x, w_\alpha\}\end{aligned}$$

and

$$b_{ijk} = \frac{\partial_y(\lambda_{\alpha,i}\lambda_{\alpha,j}) - \partial_x\lambda_{\alpha,k}}{(\lambda_{\alpha,i}-\lambda_{\alpha,k})(\lambda_{\alpha,j}-\lambda_{\alpha,k})} + \frac{\partial_y(\lambda_{\alpha,k}\lambda_{\alpha,j}) - \partial_x\lambda_{\alpha,i}}{(\lambda_{\alpha,k}-\lambda_{\alpha,i})(\lambda_{\alpha,j}-\lambda_{\alpha,i})} + \frac{\partial_y(\lambda_{\alpha,i}\lambda_{\alpha,k}) - \partial_x\lambda_{\alpha,j}}{(\lambda_{\alpha,i}-\lambda_{\alpha,j})(\lambda_{\alpha,k}-\lambda_{\alpha,j})}$$
$$= \frac{1}{\nu_\alpha y}\left[\frac{1}{f_{\alpha,1}^2}\left(2f_{\alpha,1}^2 - \varphi_\alpha f_{\alpha,2}\right) + \frac{1}{f_{\alpha,1}^3}\left(\zeta_\alpha^i + \zeta_\alpha^j + \zeta_\alpha^k\right)\left(f_{\alpha,1}^2 f_{\alpha,2} - 2\varphi_\alpha f_{\alpha,1} f_{\alpha,3} + 2\varphi_\alpha f_{\alpha,2}^2\right) w_\alpha + B_{-1} w_\alpha^2\right]$$
$$+ B_0, \quad \text{with} \quad B_{-1}, B_0 \in \mathbb{C}\{x, w_\alpha\}.$$

Since $\eta_1 = \sum_{\alpha=1,\nu_\alpha\geq 3}^{n} \sum_{1\leq i<j<k\leq\nu_\alpha} \eta_{ijk}^{\alpha\alpha\alpha}$ is a uniform and meromorphic 1-form, it follows that the principal part of the Laurent series of η_1 at $y=0$ is given by $\frac{\theta_1}{y}$, where

$$\theta_1 = \sum_{\substack{\alpha=1\\ \nu_\alpha\geq 3}}^{n} \binom{\nu_\alpha}{3}\left(-\frac{\varphi_\alpha(f_{\alpha,1}^2 - \varphi_\alpha f_{\alpha,2})}{\nu_\alpha f_{\alpha,1}^2}\mathrm{d}x + \frac{2f_{\alpha,1}^2 - \varphi_\alpha f_{\alpha,2}}{\nu_\alpha f_{\alpha,1}^2}\mathrm{d}y\right)$$
$$= \frac{1}{6}\sum_{\substack{\alpha=1\\ \nu_\alpha\geq 3}}^{n}(\nu_\alpha-1)(\nu_\alpha-2)\left(\left(1 - \frac{\varphi_\alpha f_{\alpha,2}}{f_{\alpha,1}^2}\right)(\mathrm{d}y - \varphi_\alpha \mathrm{d}x) + \mathrm{d}y\right).$$

It remains to determine the principal part of the Laurent series of η_2 at $y=0$. Again according to Lemma 2.8, we have $\eta_{ijk}^{\alpha\alpha\beta} = \tilde{a}_{ijk}(x,y)\mathrm{d}x + \tilde{b}_{ijk}(x,y)\mathrm{d}y$, where

$$\tilde{a}_{ijk} = -\frac{\left(\partial_y(\lambda_{\alpha,i}\lambda_{\alpha,j}) - \partial_x\lambda_{\beta,k}\right)\lambda_{\beta,k}}{(\lambda_{\alpha,i}-\lambda_{\beta,k})(\lambda_{\alpha,j}-\lambda_{\beta,k})} - \frac{\left(\partial_y(\lambda_{\beta,k}\lambda_{\alpha,j}) - \partial_x\lambda_{\alpha,i}\right)\lambda_{\alpha,i}}{(\lambda_{\beta,k}-\lambda_{\alpha,i})(\lambda_{\alpha,j}-\lambda_{\alpha,i})} - \frac{\left(\partial_y(\lambda_{\alpha,i}\lambda_{\beta,k}) - \partial_x\lambda_{\alpha,j}\right)\lambda_{\alpha,j}}{(\lambda_{\alpha,i}-\lambda_{\alpha,j})(\lambda_{\beta,k}-\lambda_{\alpha,j})}$$
$$= \frac{1}{\nu_\alpha y}\left[\frac{\varphi_\alpha\varphi_\beta}{\varphi_\alpha-\varphi_\beta} + \frac{\left(\zeta_\alpha^i + \zeta_\alpha^j\right)\left((\varphi_\alpha-\varphi_\beta)f_{\alpha,2} - f_{\alpha,1}^2\right)\varphi_\alpha\varphi_\beta}{(\varphi_\alpha-\varphi_\beta)^2 f_{\alpha,1}} w_\alpha + \frac{(\nu_\alpha+\nu_\beta)\zeta_\beta^k\varphi_\alpha^2 f_{\beta,1}}{\nu_\beta(\varphi_\alpha-\varphi_\beta)^2} w_\beta + \cdots\right]$$
$$+ \tilde{A}_0, \quad \text{with} \quad \tilde{A}_0 \in \mathbb{C}\{x, w_\alpha, w_\beta\}$$

and

$$\tilde{b}_{ijk} = \frac{\partial_y(\lambda_{\alpha,i}\lambda_{\alpha,j}) - \partial_x\lambda_{\beta,k}}{(\lambda_{\alpha,i}-\lambda_{\beta,k})(\lambda_{\alpha,j}-\lambda_{\beta,k})} + \frac{\partial_y(\lambda_{\beta,k}\lambda_{\alpha,j}) - \partial_x\lambda_{\alpha,i}}{(\lambda_{\beta,k}-\lambda_{\alpha,i})(\lambda_{\alpha,j}-\lambda_{\alpha,i})} + \frac{\partial_y(\lambda_{\alpha,i}\lambda_{\beta,k}) - \partial_x\lambda_{\alpha,j}}{(\lambda_{\alpha,i}-\lambda_{\alpha,j})(\lambda_{\beta,k}-\lambda_{\alpha,j})}$$
$$= -\frac{1}{\nu_\alpha y}\left[\frac{\varphi_\beta}{\varphi_\alpha-\varphi_\beta} + \frac{\left(\zeta_\alpha^i + \zeta_\alpha^j\right)\left(\varphi_\beta(\varphi_\alpha-\varphi_\beta)f_{\alpha,2} - \varphi_\alpha f_{\alpha,1}^2\right)}{(\varphi_\alpha-\varphi_\beta)^2 f_{\alpha,1}} w_\alpha + \frac{\left((2\nu_\alpha+\nu_\beta)\varphi_\alpha - \nu_\alpha\varphi_\beta\right)\zeta_\beta^k f_{\beta,1}}{\nu_\beta(\varphi_\alpha-\varphi_\beta)^2} w_\beta + \cdots\right]$$
$$+ \tilde{B}_0, \quad \text{with} \quad \tilde{B}_0 \in \mathbb{C}\{x, w_\alpha, w_\beta\}.$$

The 1-form $\eta_2 = \sum_{\alpha=1,\nu_\alpha\geq 2}^{n} \sum_{1\leq i<j\leq\nu_\alpha} \sum_{\beta=1,\beta\neq\alpha}^{n} \sum_{k=1}^{\nu_\beta} \eta_{ijk}^{\alpha\alpha\beta}$ being uniform and meromorphic, it follows that the principal part of the Laurent series of η_2 at $y=0$ is given by $\frac{\theta_2}{y}$, where

$$\theta_2 = \sum_{\substack{\alpha=1\\ \nu_\alpha\geq 2}}^{n} \binom{\nu_\alpha}{2} \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \nu_\beta\left(\frac{\varphi_\alpha\varphi_\beta}{\nu_\alpha(\varphi_\alpha-\varphi_\beta)}\mathrm{d}x - \frac{\varphi_\beta}{\nu_\alpha(\varphi_\alpha-\varphi_\beta)}\mathrm{d}y\right)$$
$$= -\frac{1}{2}\sum_{\substack{\alpha=1\\ \nu_\alpha\geq 2}}^{n}(\nu_\alpha-1)(\mathrm{d}y - \varphi_\alpha\mathrm{d}x)\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n}\frac{\nu_\beta\varphi_\beta}{\varphi_\alpha-\varphi_\beta}.$$

As a consequence, the principal part of the Laurent series of $\eta(\mathcal{W})$ at $y=0$ is given by $\frac{\theta}{y}$, where

$$\theta=\theta_1+\theta_2=\frac{1}{6}\sum_{\substack{\alpha=1\\ \nu_\alpha\geq 2}}^{n}(\nu_\alpha-1)\left\{\left[(\nu_\alpha-2)\left(1-\frac{\varphi_\alpha f_{\alpha,2}}{f_{\alpha,1}^2}\right)-3\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n}\frac{\nu_\beta\varphi_\beta}{\varphi_\alpha-\varphi_\beta}\right](\mathrm{d}y-\varphi_\alpha\mathrm{d}x)+(\nu_\alpha-2)\mathrm{d}y\right\}.$$

Thanks to (3.1), the 1-form θ can be rewritten as

$$\theta=\frac{1}{6}\sum_{\substack{\alpha=1\\ \nu_\alpha\geq 2}}^{n}(\nu_\alpha-1)\left\{\left[(\nu_\alpha-2)\left(1-\frac{\varphi_\alpha}{\nu_\alpha}\left(\frac{\partial_z\partial_y F_\alpha}{\partial_y F_\alpha}\right)\Big|_{(y,z)=(0,0)}\right)+\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n}\frac{\nu_\beta\big((\nu_\alpha-2)\varphi_\alpha-3\nu_\alpha\varphi_\beta\big)}{\nu_\alpha\big(\varphi_\alpha-\varphi_\beta\big)}\right](\mathrm{d}y-\varphi_\alpha\mathrm{d}x)+(\nu_\alpha-2)\mathrm{d}y\right\}.$$

Now, we have

$$\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n}\frac{\nu_\beta\left((\nu_\alpha-2)\varphi_\alpha-3\nu_\alpha\varphi_\beta\right)}{\nu_\alpha\left(\varphi_\alpha-\varphi_\beta\right)}=\frac{1}{\nu_\alpha}\left[(\nu_\alpha-2)(d-\nu_\alpha)-2(\nu_\alpha+1)\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n}\frac{\nu_\beta\varphi_\beta}{\varphi_\alpha-\varphi_\beta}\right],\qquad\text{because}\qquad d=\sum_{\beta=1}^{n}\nu_\beta.$$

Therefore,

$$\theta=\frac{1}{6}\sum_{\substack{\alpha=1\\ \nu_\alpha\geq 2}}^{n}(\nu_\alpha-1)\left\{\frac{1}{\nu_\alpha}\left[(\nu_\alpha-2)\left(d-\varphi_\alpha\left(\frac{\partial_z\partial_y F_\alpha}{\partial_y F_\alpha}\right)\Big|_{(y,z)=(0,0)}\right)-2(\nu_\alpha+1)\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n}\frac{\nu_\beta\varphi_\beta}{\varphi_\alpha-\varphi_\beta}\right](\mathrm{d}y-\varphi_\alpha\mathrm{d}x)+(\nu_\alpha-2)\mathrm{d}y\right\}$$

$$=\frac{1}{6}\sum_{\alpha=1}^{n}(\nu_\alpha-1)(\psi_\alpha(\mathrm{d}y-\varphi_\alpha\mathrm{d}x)+(\nu_\alpha-2)\mathrm{d}y),$$

hence the theorem follows. □

3 | HOLOMORPHY OF THE CURVATURE OF THE DUAL WEB OF A HOMOGENEOUS FOLIATION ON $\mathbb{P}^2_{\mathbb{C}}$

Following [3, Definition 2.1], a *homogeneous* foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ is given, in a suitable choice of affine coordinates (x,y), by a homogeneous 1-form $\omega=A(x,y)\mathrm{d}x+B(x,y)\mathrm{d}y$, where $A,B\in\mathbb{C}[x,y]_d$ and $\gcd(A,B)=1$.

The tangent lines to the leaves of $\mathcal{H}$ are the leaves of a d-web on the dual projective plane $\check{\mathbb{P}}^2_{\mathbb{C}}$, called the *Legendre transform* (or *dual web*) of $\mathcal{H}$, and denoted by $\mathrm{Leg}\mathcal{H}$. More precisely, let (p,q) be the affine chart of $\check{\mathbb{P}}^2_{\mathbb{C}}$ corresponding to the line $\{y=px-q\}\subset\mathbb{P}^2_{\mathbb{C}}$; then, $\mathrm{Leg}\mathcal{H}$ is given by the implicit differential equation (see [7])

$$A(x,px-q)+pB(x,px-q)=0,\qquad\text{with}\qquad x=\frac{\mathrm{d}q}{\mathrm{d}p}.\tag{4.1}$$

The Gauss map of $\mathcal{H}$ is the rational map $\mathcal{G}_{\mathcal{H}}:\mathbb{P}^2_{\mathbb{C}}\dashrightarrow\check{\mathbb{P}}^2_{\mathbb{C}}$ defined at every regular point m of $\mathcal{H}$ by $\mathcal{G}_{\mathcal{H}}(m)=\mathrm{T}^{\mathbb{P}}_m\mathcal{H}$, where $\mathrm{T}^{\mathbb{P}}_m\mathcal{H}$ denotes the tangent line to the leaf of $\mathcal{H}$ passing through m. According to [3, Lemma 3.2], the discriminant of $\mathrm{Leg}\mathcal{H}$ decomposes as

$$\Delta(\mathrm{Leg}\mathcal{H})=\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})\cup\check{\Sigma}^{\mathrm{rad}}_{\mathcal{H}}\cup\check{O},$$

where $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}$ is the transverse inflection divisor of $\mathcal{H}$, $\check{\Sigma}^{\mathrm{rad}}_{\mathcal{H}}$ is the set of lines dual to the radial singularities of $\mathcal{H}$, and finally $\check{O}$ is the dual line of the origin of the affine chart (x,y). For precise definitions of radial singularities and the inflection divisor of a foliation on $\mathbb{P}^2_{\mathbb{C}}$, we refer to [3, section 1.3].

To the homogeneous foliation $\mathcal{H}$, we can also associate the rational map $\underline{\mathcal{G}}_{\mathcal{H}} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ defined by

$$\underline{\mathcal{G}}_{\mathcal{H}}([y:x]) = [-A(x,y) : B(x,y)],$$

which allows us to completely determine the divisor $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}$ and the set $\Sigma^{\mathrm{rad}}_{\mathcal{H}}$ (see [3, section 2]):

(1) $\Sigma^{\mathrm{rad}}_{\mathcal{H}}$ consists of $[b:a:0] \in L_\infty$ such that $[a:b] \in \mathbb{P}^1_{\mathbb{C}}$ is a fixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$;
(2) $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}} = \prod_i T_i^{n_i}$, where $T_i = (b_i y - a_i x = 0)$ and $[a_i : b_i] \in \mathbb{P}^1_{\mathbb{C}}$ is a nonfixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$ of multiplicity n_i.

We know from [1, Lemma 3.1] that if the curvature of $\mathrm{Leg}\mathcal{H}$ is holomorphic on $\check{\mathbb{P}}^2_{\mathbb{C}} \setminus \check{O}$, then $\mathrm{Leg}\mathcal{H}$ is flat. The following theorem is an effective criterion for the holomorphy of the curvature of $\mathrm{Leg}\mathcal{H}$ along an irreducible component D of $\Delta(\mathrm{Leg}\mathcal{H}) \setminus \check{O}$.

Theorem 3.1. *Let $\mathcal{H}$ be a homogeneous foliation of degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$ defined by the 1-form*

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y, \quad A, B \in \mathbb{C}[x,y]_d, \quad \gcd(A,B) = 1.$$

Let (p,q) be the affine chart of $\check{\mathbb{P}}^2_{\mathbb{C}}$ associated to the line $\{y = px - q\} \subset \mathbb{P}^2_{\mathbb{C}}$ and let $D = \{p = p_0\}$ be an irreducible component of $\Delta(\mathrm{Leg}\mathcal{H}) \setminus \check{O}$. Write $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0:1]) = \{[a_1:b_1], \ldots, [a_n:b_n]\}$ and denote by ν_i the ramification index of $\underline{\mathcal{G}}_{\mathcal{H}}$ at the point $[a_i:b_i] \in \mathbb{P}^1_{\mathbb{C}}$. For $i \in \{1,\ldots,n\}$, define the polynomials $P_i \in \mathbb{C}[x,y]_{d-\nu_i}$ and $Q_i \in \mathbb{C}[x,y]_{2d-\nu_i-1}$ by

$$P_i(x,y;a_i,b_i) := \frac{\begin{vmatrix} A(x,y) & A(b_i,a_i) \\ B(x,y) & B(b_i,a_i) \end{vmatrix}}{(b_i y - a_i x)^{\nu_i}} \quad \text{and} \quad Q_i(x,y;a_i,b_i) := (\nu_i - 2)\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)P_i(x,y;a_i,b_i) + 2(\nu_i+1)\begin{vmatrix} \frac{\partial P_i}{\partial x} & A(x,y) \\ \frac{\partial P_i}{\partial y} & B(x,y) \end{vmatrix}.$$

Then, the curvature of $\mathrm{Leg}\mathcal{H}$ is holomorphic on D if and only if

$$\sum_{i=1}^{n} \left(1 - \frac{1}{\nu_i}\right) \frac{(p_0 b_i - a_i) Q_i(b_i,a_i;a_i,b_i)}{P_i(b_i,a_i;a_i,b_i) B(b_i,a_i)} = 0. \tag{4.2}$$

Remark 3.2. In particular, if $D \subset \check{\Sigma}^{\mathrm{rad}}_{\mathcal{H}} \setminus \mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$, or equivalently, if all the critical points of $\underline{\mathcal{G}}_{\mathcal{H}}$ in the fiber $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0:1])$ are fixed, then the curvature $K(\mathrm{Leg}\mathcal{H})$ is always holomorphic on D; indeed, we then have $p_0 b_i - a_i = 0$ if $\nu_i \geq 2$.

Combining this remark with [1, Lemma 3.1], we recover Theorem 3.1 of [3]: The d-web $\mathrm{Leg}\mathcal{H}$ is flat if and only if its curvature $K(\mathrm{Leg}\mathcal{H})$ is holomorphic on $\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$.

Remark 3.3. Assume that $\nu_i = \nu \geq 2$ for all $i \in \{1,\ldots,n\}$. The following assertions hold:

(1) When $\nu = 2$ (which implies that d is even), the curvature of $\mathrm{Leg}\mathcal{H}$ is always holomorphic on D.
(2) When $\nu \geq 3$, the curvature of $\mathrm{Leg}\mathcal{H}$ is holomorphic on D if and only if

$$\sum_{i=1}^{n} \frac{(p_0 b_i - a_i)\big(\partial_x B(b_i,a_i) - \partial_y A(b_i,a_i)\big)}{B(b_i,a_i)} = 0.$$

In particular, if the fiber $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0 : 1])$ contains a single nonfixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$, say $[a : b]$, then

(1) either $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0 : 1]) = \{[a : b]\}$, in which case $\nu = d$;
(2) or $\#\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0 : 1]) = 2$, in which case d is necessarily even, $d = 2k$, and $\nu = k$.

In both cases, the curvature of $\mathrm{Leg}\mathcal{H}$ is holomorphic on D if and only if the 2-form $\mathrm{d}\omega$ vanishes on the line $T = (by - ax = 0)$, which is the transverse inflection line of $\mathcal{H}$ associated to the nonfixed critical point $[a : b]$ of $\underline{\mathcal{G}}_{\mathcal{H}}$.

Example 3.4. Consider the homogeneous foliation $\mathcal{H}$ of even degree $2k \geq 4$ on $\mathbb{P}^2_{\mathbb{C}}$ defined by the 1-form

$$\omega = y^k(y-x)^k\mathrm{d}x + (y-\lambda x)^k(y-\mu x)^k\mathrm{d}y, \quad \text{where } \lambda, \mu \in \mathbb{C} \setminus \{0,1\}.$$

In the affine chart (p,q) of $\check{\mathbb{P}}^2_{\mathbb{C}}$ associated to the line $\{y = px - q\} \subset \mathbb{P}^2_{\mathbb{C}}$, the web $\mathrm{Leg}\mathcal{H}$ is implicitly described by the equation

$$(px-q)^k(px-q-x)^k + p(px-q-\lambda x)^k(px-q-\mu x)^k = 0, \qquad \text{with} \qquad x = \frac{\mathrm{d}q}{\mathrm{d}p}.$$

We see that $\mathrm{Leg}\mathcal{H}$ has a single slope $x = -q$ along $D := \{p = 0\}$, so that $D \subset \Delta(\mathrm{Leg}\mathcal{H})$. Moreover, the map $\underline{\mathcal{G}}_{\mathcal{H}}$ is given, for any $[x : y] \in \mathbb{P}^1_{\mathbb{C}}$, by

$$\underline{\mathcal{G}}_{\mathcal{H}}([x : y]) = [-x^k(x-y)^k : (x-\lambda y)^k(x-\mu y)^k].$$

In particular, the fiber $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([0 : 1])$ consists of the two points $[0 : 1]$ and $[1 : 1]$: The point $[0 : 1]$ (resp. $[1 : 1]$) is critical and fixed (resp. nonfixed) for $\underline{\mathcal{G}}_{\mathcal{H}}$ of multiplicity $k - 1$. From Remark 3.3, we deduce the following:

(1) If $k = 2$, then the curvature of $\mathrm{Leg}\mathcal{H}$ is holomorphic on D.
(2) If $k > 2$, then the curvature of $\mathrm{Leg}\mathcal{H}$ is holomorphic on D if and only if

$$0 \equiv \mathrm{d}\omega\Big|_{y=x} = -k(\lambda-1)^{k-1}(\mu-1)^{k-1}x^{2k-1}(\lambda+\mu-2\lambda\mu)\mathrm{d}x \wedge \mathrm{d}y,$$

that is, if and only if λ and μ satisfy the equation $\lambda + \mu - 2\lambda\mu = 0$.

To prove Theorem 3.1, we need the following lemma.

Lemma 3.5. *Let $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be a rational map of degree d; $f(z) = \dfrac{a(z)}{b(z)}$ where a and b are polynomials without common factor and* $\max(\deg a, \deg b) = d$. *Let $w_0 \in \mathbb{C}$ and write $f^{-1}(w_0) = \{z_1, z_2, \ldots, z_n\}$. Suppose that $z_i \neq \infty$ for all $i \in \{1, \ldots, n\}$ and let ν_i denote the ramification index of f at the point z_i. Then, there exists $c \in \mathbb{C}^*$ such that $a(z) = w_0 b(z) + c\prod_{i=1}^{n}(z - z_i)^{\nu_i}$.*

Proof. According to [3, Lemma 3.9], for every $i \in \{1, \ldots, n\}$, there exists a polynomial $\phi_i \in \mathbb{C}[z]$ of degree $\leq d - \nu_i$ satisfying $\phi_i(z_i) \neq 0$ and such that $a(z) = w_0 b(z) + \phi_i(z)(z - z_i)^{\nu_i}$. This implies that for all $i, j \in \{1, \ldots, n\}$, $\phi_i(z)(z - z_i)^{\nu_i} = \phi_j(z)(z - z_j)^{\nu_j}$, so that for any $j \neq i$, $(z - z_j)^{\nu_j}$ divides ϕ_i. As a result, $\phi_i \in \mathbb{C}[z]$ has degree $d - \nu_i$ and writes as $\phi_i(z) = c\displaystyle\prod_{j=1, j\neq i}^{n}(z - z_j)^{\nu_j}$ for some $c \in \mathbb{C}^*$, hence the statement is proved. □

Proof of Theorem 3.1. Let $\delta \in \mathbb{C}$ be such that $b_i - a_i\delta \neq 0$ for all $i = 1, \ldots, n$. Up to conjugating ω by the linear transformation $(x + \delta y, y)$, we can assume that none of the lines $L_i = (b_i y - a_i x = 0)$ are vertical, that is, $b_i \neq 0$ for all $i = 1, \ldots, n$. Setting $r_i := \frac{a_i}{b_i}$, we have $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}(p_0) = \{r_1, \ldots, r_n\}$ with $\underline{\mathcal{G}}_{\mathcal{H}}(z) = -\dfrac{A(1,z)}{B(1,z)}$. According to Lemma 3.5, there exists a constant

$c \in \mathbb{C}^*$ such that

$$-A(1,z) = p_0 B(1,z) - c \prod_{i=1}^{n} (z-r_i)^{\nu_i}.$$

Moreover, the d-web Leg$\mathcal{H}$ is given by Equation (4.1); since $A, B \in \mathbb{C}[x,y]_d$, this equation can then be rewritten as

$$0 = x^d \left[A\left(1, p-\frac{q}{x}\right) + pB\left(1, p-\frac{q}{x}\right)\right] = x^d \left[(p-p_0)B\left(1, p-\frac{q}{x}\right) + c\prod_{i=1}^{n}(p-\frac{q}{x}-r_i)\nu^i\right], \quad \text{with} \quad x = \frac{\mathrm{d}q}{\mathrm{d}p}.$$

Set $\check{x} := q$, $\check{y} := p - p_0$, and $\check{p} := \dfrac{\mathrm{d}\check{y}}{\mathrm{d}\check{x}} = \dfrac{1}{x}$; in these new coordinates, $D = \{\check{y} = 0\}$ and Leg$\mathcal{H}$ is described by the differential equation

$$F(\check{x},\check{y},\check{p}) := \check{y}B(1, \check{y}+p_0-\check{p}\check{x}) + c\prod_{i=1}^{n}(\check{y}+p_0-\check{p}\check{x}-r_i)^{\nu_i} = 0.$$

We have $F(\check{x},0,\check{p}) = c(-\check{x})^d \prod_{i=1}^{n}(\check{p}-\varphi_i(\check{x}))^{\nu_i}$, where $\varphi_i(\check{x}) = \frac{p_0-r_i}{\check{x}}$. Note that if $\nu_i \geq 2$, then $\partial_{\check{y}}F(\check{x},0,\varphi_i(\check{x})) = B(1,r_i) \neq 0$; since $\partial_{\check{p}}F(\check{x},0,\varphi_i(\check{x})) \not\equiv 0$ if $\nu_i = 1$, it follows that the surface $S_{\mathrm{Leg}\mathcal{H}}$ is smooth along $D = \{\check{y}=0\}$. Furthermore, if $\nu_i \geq 3$, then $\partial_{\check{p}}\partial_{\check{y}}F(\check{x},0,\varphi_i(\check{x})) = -\check{x}\partial_y B(1,r_i)$. Thus, by Theorem 3.1, the curvature of Leg$\mathcal{H}$ is holomorphic on $D = \{\check{y}=0\}$ if and only if $\sum_{i=1}^{n}(\nu_i-1)\varphi_i(\check{x})\psi_i \equiv 0$ where, for all $i \in \{1,\dots,n\}$ such that $\nu_i \geq 2$,

$$\psi_i = \frac{1}{\nu_i}\left[(\nu_i-2)\left(d + (p_0-r_i)\frac{\partial_y B(1,r_i)}{B(1,r_i)}\right) + 2(\nu_i+1)\sum_{j=1, j\neq i}^{n}\frac{\nu_j(p_0-r_j)}{r_i-r_j}\right].$$

We note that

$$\sum_{j=1,j\neq i}^{n}\frac{\nu_j(p_0-r_j)}{r_i-r_j} = d - \nu_i + (p_0-r_i)\sum_{j=1,j\neq i}^{n}\frac{\nu_j}{r_i-r_j} = d-\nu_i+(p_0-r_i)\frac{f_i'(r_i)}{f_i(r_i)},$$

where $f_i(z) := c\prod_{j=1,j\neq i}^{n}(z-r_j)^{\nu_j} = \dfrac{A(1,z)+p_0B(1,z)}{(z-r_i)^{\nu_i}} = \dfrac{P_i(1,z;r_i,1)}{B(1,r_i)}$. Therefore,

$$\sum_{j=1,j\neq i}^{n}\frac{\nu_j(p_0-r_j)}{r_i-r_j} = d-\nu_i+(p_0-r_i)\frac{\partial_y P_i(1,r_i;r_i,1)}{P_i(1,r_i;r_i,1)} = \frac{\begin{vmatrix}\partial_x P_i(1,r_i;r_i,1) & A(1,r_i)\\ \partial_y P_i(1,r_i;r_i,1) & B(1,r_i)\end{vmatrix}}{B(1,r_i)P_i(1,r_i;r_i,1)},$$

because $p_0 = \underline{\mathcal{G}}_{\mathcal{H}}(r_i) = -\dfrac{A(1,r_i)}{B(1,r_i)}$ and $(d-\nu_i)P_i(1,r_i;r_i,1) = \partial_x P_i(1,r_i;r_i,1) + r_i\partial_y P_i(1,r_i;r_i,1)$ (Euler's identity).

On the other hand, let us fix $i \in \{1,\dots,n\}$ such that $\nu_i \geq 2$; from the equalities $p_0 = \underline{\mathcal{G}}_{\mathcal{H}}(r_i)$ and $\underline{\mathcal{G}}_{\mathcal{H}}'(r_i) = 0$, we deduce that $p_0\partial_y B(1,r_i) = -\partial_y A(1,r_i)$, so that

$$dB(1,r_i) + (p_0-r_i)\partial_y B(1,r_i) = dB(1,r_i) - r_i\partial_y B(1,r_i) - \partial_y A(1,r_i) = \partial_x B(1,r_i) - \partial_y A(1,r_i),$$

thanks to Euler's identity.

It follows that for all $i \in \{1,\dots,n\}$ such that $\nu_i \geq 2$, $\psi_i = \dfrac{Q_i(1,r_i;r_i,1)}{\nu_i P_i(1,r_i;r_i,1)B(1,r_i)}$. As a consequence, $K(\mathrm{Leg}\mathcal{H})$ is holomorphic on $D = \{\check{y}=0\}$ if and only if

$$\frac{1}{\check{x}}\sum_{i=1}^{n}\left(1-\frac{1}{\nu_i}\right)\frac{(p_0-r_i)Q_i(1,r_i;r_i,1)}{P_i(1,r_i;r_i,1)B(1,r_i)} \equiv 0,$$

which ends the proof of the theorem. □

Corollary 3.6. *Let $\mathcal{H}$ be a homogeneous foliation of degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$ defined by the 1-form*

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y, \quad A, B \in \mathbb{C}[x,y]_d, \quad \gcd(A,B) = 1.$$

Assume that $\mathcal{H}$ possesses a transverse inflection line $T = (ax + by = 0)$ of order $\nu - 1$. Suppose moreover that $[-a : b] \in \mathbb{P}^1_{\mathbb{C}}$ is the only nonfixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$ in its fiber $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}(\underline{\mathcal{G}}_{\mathcal{H}}([-a : b]))$. Then, the curvature of $\mathrm{Leg}\mathcal{H}$ is holomorphic on $T' = \mathcal{G}_{\mathcal{H}}(T)$ if and only if $Q(b, -a; a, b) = 0$, where

$$Q(x,y;a,b) := (\nu-2)\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)P(x,y;a,b) + 2(\nu+1)\begin{vmatrix} \frac{\partial P}{\partial x} & A(x,y) \\ \frac{\partial P}{\partial y} & B(x,y) \end{vmatrix} \quad \text{and} \quad P(x,y;a,b) := \frac{\begin{vmatrix} A(x,y) & A(b,-a) \\ B(x,y) & B(b,-a) \end{vmatrix}}{(ax+by)^{\nu}}.$$

Remark 3.7. When the line $T = (ax + by = 0)$ is of minimal inflection order 1 (i.e., if $\nu = 2$) and under the more restrictive hypothesis that the point $[-a : b]$ is the only critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$ in its fiber, we recover [3, Theorem 3.5]. When T is of maximal inflection order $d - 1$ (i.e., if $\nu = d$), we recover [3, Theorem 3.8].

Proof. Up to linear conjugation, we can assume that T' is not the line at infinity of $\check{\mathbb{P}}^2_{\mathbb{C}}$; then, T' has the equation $p = p_0$, where $p_0 = -\frac{A(b,-a)}{B(b,-a)}$. Write $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0 : 1]) = \{[a_1 : b_1], \ldots, [a_n : b_n]\}$ with $[a_1 : b_1] = [-a : b]$. Denoting by ν_i the ramification index of $\underline{\mathcal{G}}_{\mathcal{H}}$ at the point $[a_i, b_i]$, we have $\nu_1 = \nu$ and, by application of Theorem 3.1, the holomorphy of $K(\mathrm{Leg}\mathcal{H})$ along T' is characterized by Equation (4.2). Now, the point $[a_1 : b_1]$ being not fixed by $\underline{\mathcal{G}}_{\mathcal{H}}$, we have $p_0 b_1 - a_1 \neq 0$. Moreover, the hypothesis that $[a_1 : b_1]$ is the only nonfixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$ in the fiber $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0 : 1])$ ensures that $p_0 b_i - a_i = 0$ for all $i \in \{2, \ldots, n\}$ such that $\nu_i \geq 2$. It follows that $K(\mathrm{Leg}\mathcal{H})$ is holomorphic on T' if and only if $0 = Q_1(b_1, a_1; a_1, b_1) = Q(b, -a; a, b)$. Hence, the corollary is proved. □

4 | GALOIS HOMOGENEOUS FOLIATIONS HAVING A FLAT LEGENDRE TRANSFORM

Following [2, Definition 6.16] a foliation $\mathcal{F}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ is said to be Galois if there is a Zariski open subset U of $\mathbb{P}^2_{\mathbb{C}}$ such that the Gauss map $\mathcal{G}_{\mathcal{F}} : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \check{\mathbb{P}}^2_{\mathbb{C}}$, defined by $m \notin \mathrm{Sing}\mathcal{F} \mapsto \mathrm{T}^{\mathbb{P}}_m\mathcal{F}$, induces a Galois covering from U onto $\mathcal{G}_{\mathcal{F}}(U)$, necessarily of degree d. This is equivalent to the existence of a subgroup G of order d of the group $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ of birational transformations of $\mathbb{P}^2_{\mathbb{C}}$ such that for all $\gamma \in G$, we have $\mathcal{G}_{\mathcal{F}} \circ \gamma = \mathcal{G}_{\mathcal{F}}$.

In particular, if $\mathcal{F}$ is homogeneous, then its associated map $\underline{\mathcal{G}}_{\mathcal{F}} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is a ramified covering of degree d. Moreover, $\mathcal{F}$ is Galois if and only if $\underline{\mathcal{G}}_{\mathcal{F}}$ is Galois [2, Proposition 6.19], or equivalently, if and only if $\underline{\mathcal{G}}_{\mathcal{F}}$ has the same ramification indices at all the points of the same fiber [2, Theorem A].

Let us note that if $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is Galois, then $\ell \circ f \circ \rho : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is also Galois for any ℓ and ρ belonging to the automorphism group $\mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$. Recall the following result, due to Klein [6, Part I, Chapter II] (see also [2, Theorem 4.18]), classifying the ramified Galois coverings $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ up to the left–right action $f \mapsto \ell \circ f \circ \rho$, where $(\ell, \rho) \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}}) \times \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$.

Theorem 4.1. *Let $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be a ramified Galois covering of degree d. Up to the left–right action of $\mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}}) \times \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$, f is of one of the following types:*

1. $f_1 = z^d$;
2. $f_2 = \frac{(z^k+1)^2}{4z^k}$ *if d is even, $d = 2k$;*
3. $f_3 = \left(\frac{z^4+2\mathrm{i}\sqrt{3}z^2+1}{z^4-2\mathrm{i}\sqrt{3}z^2+1}\right)^3$ *if $d = 12$;*

4. $f_4 = \frac{(z^8+14z^4+1)^3}{108z^4(z^4-1)^4}$ *if* $d = 24$;
5. $f_5 = \frac{(z^{20}-228z^{15}+494z^{10}+228z^5+1)^3}{-1728z^5(z^{10}+11z^5-1)^5}$ *if* $d = 60$.

Moreover, the Galois group of f *is cyclic if and only if* f *is left–right conjugate to* f_1.

Definition 4.2. Let $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be a rational map of degree d. We call *associated foliation* to f the homogeneous foliation $\mathscr{H}(f)$ of $\mathbb{P}^2_{\mathbb{C}}$ whose associated rational map $\underline{\mathcal{G}}_{\mathscr{H}(f)}$ is precisely f.

Note that if f is defined by $f([x : y]) = [A(x,y) : B(x,y)]$, where $A, B \in \mathbb{C}[x,y]_d$ and $\gcd(A,B) = 1$, then $\mathscr{H}(f)$ is given by the 1-form $\omega = A(y,x)\mathrm{d}x - B(y,x)\mathrm{d}y$.

According to [2, Proposition 6.19], Theorem 4.1 translates in terms of homogeneous foliations as follows:

Theorem 4.3. *Let* $\mathcal{H}$ *be a Galois homogeneous foliation on* $\mathbb{P}^2_{\mathbb{C}}$. *Then, there exist* $i \in \{1, \ldots, 5\}$ *and* $\ell, \rho \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ *such that* $\mathcal{H} = \mathscr{H}(\ell \circ f_i \circ \rho)$.

The following theorem is the main result of this section.

Theorem 4.4. *Let* $\mathcal{H}$ *be a Galois homogeneous foliation of degree* $d \geq 3$ *on* $\mathbb{P}^2_{\mathbb{C}}$. *Denote by* $\mathrm{Gal}(\underline{\mathcal{G}}_{\mathcal{H}})$ *the Galois group of the covering* $\underline{\mathcal{G}}_{\mathcal{H}}$. *We have the following dichotomy:*

(1) If $\mathrm{Gal}(\underline{\mathcal{G}}_{\mathcal{H}})$ *is cyclic, then the* d*-web* $\mathrm{Leg}\mathcal{H}$ *is flat if and only if* $\mathcal{H}$ *is linearly conjugate to one of the two foliations* $\mathcal{H}_1^d$ *and* $\mathcal{H}_2^d$ *defined, respectively, by the 1-forms*

$$\omega_1^d = y^d\mathrm{d}x - x^d\mathrm{d}y \qquad \textit{and} \qquad \omega_2^d = x^d\mathrm{d}x - y^d\mathrm{d}y.$$

(2) If $\mathrm{Gal}(\underline{\mathcal{G}}_{\mathcal{H}})$ *is noncyclic, then the* d*-web* $\mathrm{Leg}\mathcal{H}$ *is flat.*

To prove this theorem, we need the following lemma.

Lemma 4.5. *Let* $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ *be a rational map of degree* d *defined, for any* $[x : y] \in \mathbb{P}^1_{\mathbb{C}}$, *by*

$$f([x : y]) = [A(x,y) : B(x,y)], \quad A, B \in \mathbb{C}[x,y]_d, \quad \gcd(A,B) = 1.$$

Let $p_0 \in \mathbb{C} \cup \{\infty\}$ *be a critical value of* f *and write* $f^{-1}(p_0) = \{[a_1 : b_1], \ldots, [a_n : b_n]\}$. *Suppose that the ramification indices of* f *at the points* $[a_i : b_i]$ *are all equal to each other and let* ν *be their common value. For* $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$, *denote by* $\mathcal{H}_h = \mathscr{H}(h \circ f)$ *the homogeneous foliation associated to the rational map* $h \circ f$. *Let* (p,q) *be the affine chart of* $\check{\mathbb{P}}^2_{\mathbb{C}}$ *corresponding to the line* $\{y = px - q\} \subset \mathbb{P}^2_{\mathbb{C}}$ *and let* $D_h := \{p = h(p_0)\} \subset \Delta(\mathrm{Leg}\mathcal{H}_h)$.

(1) If $\nu = 2$, *then the curvature of* $\mathrm{Leg}\mathcal{H}_h$ *is holomorphic on* D_h *for all* $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$.
(2) If $\nu \geq 3$ *and* $p_0 \in \mathbb{C}$, *then the curvature of* $\mathrm{Leg}\mathcal{H}_h$ *is holomorphic on* D_h *for all* $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ *if and only if*

$$\sum_{i=1}^{n} \frac{b_i\partial_x B(a_i,b_i)}{B(a_i,b_i)} = 0, \quad \sum_{i=1}^{n} \frac{b_i\partial_y B(a_i,b_i) - a_i\partial_x B(a_i,b_i)}{B(a_i,b_i)} = 0, \quad \sum_{i=1}^{n} \frac{a_i\partial_y B(a_i,b_i)}{B(a_i,b_i)} = 0. \tag{5.1}$$

(3) If $\nu \geq 3$ *and* $p_0 = \infty$, *then the curvature of* $\mathrm{Leg}\mathcal{H}_h$ *is holomorphic on* D_h *for all* $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ *if and only if*

$$\sum_{i=1}^{n} \frac{b_i\partial_x A(a_i,b_i)}{A(a_i,b_i)} = 0, \quad \sum_{i=1}^{n} \frac{b_i\partial_y A(a_i,b_i) - a_i\partial_x A(a_i,b_i)}{A(a_i,b_i)} = 0, \quad \sum_{i=1}^{n} \frac{a_i\partial_y A(a_i,b_i)}{A(a_i,b_i)} = 0. \tag{5.2}$$

Proof. Let $h : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be an automorphism of $\mathbb{P}^1_{\mathbb{C}}$; $h(z) = \dfrac{\alpha z+\beta}{\gamma z+\delta}$, where $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ with $\alpha\delta-\beta\gamma \neq 0$. Then, the foliation $\mathcal{H}_h$ is given by

$$\omega_h = (\alpha\, A(y,x)+\beta\, B(y,x))\mathrm{d}x - (\gamma\, A(y,x)+\delta\, B(y,x))\mathrm{d}y.$$

Moreover, we have

$$(h\circ f)^{-1}(h(p_0)) = f^{-1}(p_0) = \{[a_1 : b_1], \dots, [a_n : b_n]\};$$

since by hypothesis the ramification indices of f at the points $[a_i : b_i]$ are all equal to each other and equal to ν, the same is true for the ramification indices of $h\circ f$ at these points, because $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$. According to Remark 3.3, it follows that:

i. If $\nu = 2$, then $K(\mathrm{Leg}\mathcal{H}_h)$ is holomorphic on D_h for all $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$.
ii. If $\nu \geq 3$, then $K(\mathrm{Leg}\mathcal{H}_h)$ is holomorphic on D_h for all $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ if and only if

$$\sum_{i=1}^{n} \frac{(h(p_0)b_i - a_i)\big(\alpha\partial_x A(a_i,b_i)+\beta\partial_x B(a_i,b_i)+\gamma\partial_y A(a_i,b_i)+\delta\partial_y B(a_i,b_i)\big)}{\gamma\, A(a_i,b_i)+\delta\, B(a_i,b_i)} = 0. \tag{5.3}$$

ii.1. If $p_0 \in \mathbb{C}$, then, from $f([a_i : b_i]) = [p_0 : 1]$ and the fact that $[a_i : b_i]$ are critical points of f, we deduce the equalities $A(a_i,b_i) = p_0 B(a_i,b_i)$, $\partial_x A(a_i,b_i) = p_0\partial_x B(a_i,b_i)$, and $\partial_y A(a_i,b_i) = p_0\partial_y B(a_i,b_i)$, so that (5.3) can be rewritten as

$$h(p_0)^2\sum_{i=1}^{n}\frac{b_i\partial_x B(a_i,b_i)}{B(a_i,b_i)} + h(p_0)\sum_{i=1}^{n}\frac{b_i\partial_y B(a_i,b_i)-a_i\partial_x B(a_i,b_i)}{B(a_i,b_i)} - \sum_{i=1}^{n}\frac{a_i\partial_y B(a_i,b_i)}{B(a_i,b_i)} = 0.$$

As a result, $K(\mathrm{Leg}\mathcal{H}_h)$ is holomorphic on D_h for all $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ if and only if the system (5.1) is satisfied.

ii.2. If $p_0 = \infty$, then $B(a_i,b_i) = \partial_x B(a_i,b_i) = \partial_y B(a_i,b_i) = 0$ and (5.3) becomes

$$h(p_0)^2\sum_{i=1}^{n}\frac{b_i\partial_x A(a_i,b_i)}{A(a_i,b_i)} + h(p_0)\sum_{i=1}^{n}\frac{b_i\partial_y A(a_i,b_i)-a_i\partial_x A(a_i,b_i)}{A(a_i,b_i)} - \sum_{i=1}^{n}\frac{a_i\partial_y A(a_i,b_i)}{A(a_i,b_i)} = 0.$$

As a consequence, $K(\mathrm{Leg}\mathcal{H}_h)$ is holomorphic on D_h for all $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ if and only if the system (5.2) is satisfied.

Hence, the lemma is proved. □

Proof of Theorem 4.4. i. Suppose that $\mathrm{Gal}(\underline{\mathcal{G}}_{\mathcal{H}})$ is cyclic. Then, by Theorem 4.1, $\underline{\mathcal{G}}_{\mathcal{H}}$ is left–right conjugate to $f_1 = z^d$. Since f_1 has exactly two critical points (namely 0 and ∞), the same is true for $\underline{\mathcal{G}}_{\mathcal{H}}$. This implies, according to [3, Proposition 4.1], that the d-web $\mathrm{Leg}\mathcal{H}$ is flat if and only if $\mathcal{H}$ is linearly conjugate to one of the two foliations $\mathcal{H}_1^d, \mathcal{H}_2^d$.

ii. Suppose that $\mathrm{Gal}(\underline{\mathcal{G}}_{\mathcal{H}})$ is noncyclic. According to Theorem 4.1, there exist $i \in \{2,\dots,5\}$ and $\ell,\rho \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ such that $\underline{\mathcal{G}}_{\mathcal{H}} = \ell\circ f_i\circ\rho$ and therefore $\mathcal{H} = \mathscr{H}(\ell\circ f_i\circ\rho)$. In particular, there exist $i \in \{2,\dots,5\}$ and $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ such that $\mathcal{H}$ is linearly conjugate to the foliation $\mathcal{H}_h^{(i)} := \mathscr{H}(h\circ f_i)$; indeed, it suffices to take $h = \rho\circ\ell$, because $h\circ f_i = \rho\circ(\ell\circ f_i\circ\rho)\circ\rho^{-1}$. To show that the d-web $\mathrm{Leg}\mathcal{H}$ is flat, it suffices therefore to show that for all $i \in \{2,\dots,5\}$ and all $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$, the d-web $\mathrm{Leg}\mathcal{H}_h^{(i)}$ is flat. Now, for all $i \in \{2,\dots,5\}$, the map f_i being a ramified Galois covering of $\mathbb{P}^1_{\mathbb{C}}$ by itself, [2, Theorem A] implies that the ramification indices of f_i at the points of the same fiber $f_i^{-1}(p_0)$ have the same value, which we will denote by $\nu(f_i,p_0)$. Thanks to [3, Theorem 3.1], it suffices again to apply Lemma 4.5 to each of the f_i and to show that for every critical value $p_0 \in \mathbb{P}^1_{\mathbb{C}}$ of f_i, the curvature of $\mathrm{Leg}\mathcal{H}_h^{(i)}$ is holomorphic on the component $D_h^{(i)}(p_0) := \{p = h(p_0)\}$ of $\Delta(\mathrm{Leg}\mathcal{H}_h^{(i)})$ for all $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$.

First of all, a straightforward computation shows that each of the $f_i, i = 2,\dots,5$, has as critical values 0, 1, and ∞.

The case of the critical value $p_0 = 1$ is immediate. Indeed, it is easy to verify that for all $i \in \{2,\dots,5\}$, $\nu(f_i,1) = 2$, so that the curvature of $\mathrm{Leg}\mathcal{H}_h^{(i)}$ is holomorphic on $D_h^{(i)}(1)$ for all $i \in \{2,\dots,5\}$ and all $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ (Lemma 4.5).

The case where $i = 2$ and $p_0 = 0$ is also immediate. Indeed, we have $\nu(f_2, 0) = 2$, which implies that $K(\mathrm{Leg}\mathcal{H}_h^{(2)})$ is holomorphic on $D_h^{(2)}(0)$ for all $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$.

Let us consider the case where $i = 2$ and $p_0 = \infty$. The map f_2 is defined in homogeneous coordinates by

$$f_2 : [x : y] \mapsto [A_2(x,y) : B_2(x,y)], \quad \text{where } A_2(x,y) = (x^k + y^k)^2 \text{and } B_2(x,y) = 4x^k y^k.$$

Moreover, the fiber $f_2^{-1}(\infty)$ consists of the two points $0 = [0 : 1]$ and $\infty = [1 : 0]$; in particular, $\nu(f_2, \infty) = k$. If $k = 2$, then $K(\mathrm{Leg}\mathcal{H}_h^{(2)})$ is holomorphic on $D_h^{(2)}(\infty)$ for all $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$. Suppose $k \geq 3$. We have

$$\sum_{[a:b]\in f_2^{-1}(\infty)} \frac{b\partial_x A_2(a,b)}{A_2(a,b)} = \frac{\partial_x A_2(0,1)}{A_2(0,1)} = 0, \qquad \sum_{[a:b]\in f_2^{-1}(\infty)} \frac{b\partial_y A_2(a,b) - a\partial_x A_2(a,b)}{A_2(a,b)} = \frac{\partial_y A_2(0,1)}{A_2(0,1)} - \frac{\partial_x A_2(1,0)}{A_2(1,0)} = 0,$$
$$\sum_{[a:b]\in f_2^{-1}(\infty)} \frac{a\partial_y A_2(a,b)}{A_2(a,b)} = \frac{\partial_y A_2(1,0)}{A_2(1,0)} = 0;$$

it follows, by Lemma 4.5, that $K(\mathrm{Leg}\mathcal{H}_h^{(2)})$ is holomorphic on $D_h^{(2)}(\infty)$ for all $h \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$.

Let us study the case where $i = 5$ and $p_0 = 0$. Consider the polynomials

$$P(w) = w^4 - 228w^3 + 494w^2 + 228w + 1 \qquad \text{and} \qquad Q(w) = -\sqrt[5]{1728}(w^2 + 11w - 1);$$

the map f_5 is given, for any $[x : y] \in \mathbb{P}^1_{\mathbb{C}}$, by $f_5([x : y]) = [A_5(x,y) : B_5(x,y)]$, where

$$A_5(x,y) = \left(y^{20} P\left(\frac{x^5}{y^5}\right)\right)^3 \qquad \text{and} \qquad B_5(x,y) = \left(xy^{11} Q\left(\frac{x^5}{y^5}\right)\right)^5.$$

The polynomial $P(w)$ has as roots the real numbers

$$w_1 = 57 - 25\sqrt{5} + 5\sqrt{255 - 114\sqrt{5}}, \quad w_2 = 57 - 25\sqrt{5} - 5\sqrt{255 - 114\sqrt{5}}, \quad w_3 = 57 + 25\sqrt{5} + 5\sqrt{255 + 114\sqrt{5}},$$
$$w_4 = 57 + 25\sqrt{5} - 5\sqrt{255 + 114\sqrt{5}};$$

by setting $\zeta = \exp(\frac{2i\pi}{5})$ and $u_j = \sqrt[5]{w_j} \in \mathbb{R}$, $j = 1, \dots, 4$, we have

$$f_5^{-1}(0) = \left\{[\zeta^l u_j : 1] \mid j = 1, \dots, 4, l = 0, \dots, 4\right\}.$$

In particular, $f_5^{-1}(0)$ has cardinality 20 and therefore $\nu(f_5, 0) = 60/20 = 3$. Furthermore, by a straightforward computation, we obtain the following equalities:

$$\frac{b\partial_x B_5(a,b)}{B_5(a,b)}\Big|_{(a,b)=(\zeta^l u_j,1)} = 5\zeta^{5-l}\left(\frac{1}{u_j} + \frac{5w_j Q'(w_j)}{u_j Q(w_j)}\right), \quad \frac{a\partial_y B_5(a,b)}{B_5(a,b)}\Big|_{(a,b)=(\zeta^l u_j,1)} = 5\zeta^l u_j\left(11 - \frac{5w_j Q'(w_j)}{Q(w_j)}\right),$$
$$\frac{b\partial_y B_5(a,b) - a\partial_x B_5(a,b)}{B_5(a,b)}\Big|_{(a,b)=(\zeta^l u_j,1)} = g(w_j),$$

where $g : x \mapsto -\frac{50(x^2+1)}{x^2+11x-1}$, so that

$$\sum_{j=1}^{4}\sum_{l=0}^{4}\frac{b\partial_x B_5(a,b)}{B_5(a,b)}\Big|_{(a,b)=(\zeta^l u_j,1)}=0,\qquad \sum_{j=1}^{4}\sum_{l=0}^{4}\frac{b\partial_y B_5(a,b)-a\partial_x B_5(a,b)}{B_5(a,b)}\Big|_{(a,b)=(\zeta^l u_j,1)}=0,$$

$$\sum_{j=1}^{4}\sum_{l=0}^{4}\frac{a\partial_y B_5(a,b)}{B_5(a,b)}\Big|_{(a,b)=(\zeta^l u_j,1)}=0,$$

because $\sum_{l=0}^{4}\zeta^l=\sum_{l=0}^{4}\zeta^{5-l}=0$ and $\sum_{j=1}^{4}g(w_j)=0$. Thus, we deduce from Lemma 4.5 that $K(\mathrm{Leg}\mathcal{H}_h^{(5)})$ is holomorphic on $D_h^{(5)}(0)$ for all $h\in\mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$.

Let us examine the case where $i=5$ and $p_0=\infty$. Set $\widetilde{w}_1=\frac{-11+5\sqrt{5}}{2}$, $\widetilde{w}_2=\frac{-11-5\sqrt{5}}{2}$, $\widetilde{u}_1=\frac{-1+\sqrt{5}}{2}$, and $\widetilde{u}_2=\frac{-1-\sqrt{5}}{2}$ (the $\widetilde{w}_j$ are the two roots of $Q(w)$ and $\widetilde{u}_j=\sqrt[5]{\widetilde{w}_j}$). Then,

$$f_5^{-1}(\infty)=\Big\{[0:1],\,[1:0],\,[\zeta^l\widetilde{u}_j:1]\ \big|\ j=1,2,l=0,\dots,4\Big\};$$

in particular, $\#f_5^{-1}(\infty)=12$ and consequently $\nu(f_5,\infty)=60/12=5$. Moreover, a straightforward computation leads to

$$\frac{b\partial_x A_5(a,b)}{A_5(a,b)}\Big|_{(a,b)=(0,1)}=0,\quad \frac{a\partial_y A_5(a,b)}{A_5(a,b)}\Big|_{(a,b)=(0,1)}=0,\quad \frac{b\partial_y A_5(a,b)-a\partial_x A_5(a,b)}{A_5(a,b)}\Big|_{(a,b)=(0,1)}=60,$$

$$\frac{b\partial_x A_5(a,b)}{A_5(a,b)}\Big|_{(a,b)=(1,0)}=0,\quad \frac{a\partial_y A_5(a,b)}{A_5(a,b)}\Big|_{(a,b)=(1,0)}=0,\quad \frac{b\partial_y A_5(a,b)-a\partial_x A_5(a,b)}{A_5(a,b)}\Big|_{(a,b)=(1,0)}=-60,$$

$$\frac{b\partial_x A_5(a,b)}{A_5(a,b)}\Big|_{(a,b)=(\zeta^l\widetilde{u}_j,1)}=\frac{15\zeta^{5-l}\widetilde{w}_jP'(\widetilde{w}_j)}{\widetilde{u}_jP(\widetilde{w}_j)},\quad \frac{a\partial_y A_5(a,b)}{A_5(a,b)}\Big|_{(a,b)=(\zeta^l\widetilde{u}_j,1)}=15\zeta^l\widetilde{u}_j\left(4-\frac{\widetilde{w}_jP'(\widetilde{w}_j)}{P(\widetilde{w}_j)}\right),$$

$$\frac{b\partial_y A_5(a,b)-a\partial_x A_5(a,b)}{A_5(a,b)}\Big|_{(a,b)=(\zeta^l\widetilde{u}_j,1)}=\widetilde{g}(\widetilde{w}_j),$$

where $\widetilde{g}:x\mapsto-\frac{60(x^4-114x^3-114x-1)}{x^4-228x^3+494x^2+228x+1}$. Therefore, we have

$$\sum_{[a:b]\in f_5^{-1}(\infty)}\frac{b\partial_x A_5(a,b)}{A_5(a,b)}=\sum_{j=1}^{2}\frac{15\widetilde{w}_jP'(\widetilde{w}_j)}{\widetilde{u}_jP(\widetilde{w}_j)}\sum_{l=0}^{4}\zeta^{5-l}=0,\qquad \sum_{[a:b]\in f_5^{-1}(\infty)}\frac{a\partial_y A_5(a,b)}{A_5(a,b)}=\sum_{j=1}^{2}15\widetilde{u}_j\left(4-\frac{\widetilde{w}_jP'(\widetilde{w}_j)}{P(\widetilde{w}_j)}\right)\sum_{l=0}^{4}\zeta^l=0,$$

$$\sum_{[a:b]\in f_5^{-1}(\infty)}\frac{b\partial_y A_5(a,b)-a\partial_x A_5(a,b)}{A_5(a,b)}=5\sum_{j=1}^{2}\widetilde{g}(\widetilde{w}_j)=0.$$

According to Lemma 4.5, it follows that $K(\mathrm{Leg}\mathcal{H}_h^{(5)})$ is holomorphic on $D_h^{(5)}(\infty)$ for all $h\in\mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$.
The remaining cases (those where $i\in\{3,4\}$ and $p_0\in\{0,\infty\}$) are treated similarly. □

Remark 4.6. For $d\geq 3$, denote by **FP**(d) the algebraic set consisting of foliations of degree d on $\mathbb{P}^2_{\mathbb{C}}$ with a flat Legendre transform. In [4, Theorem D], we showed that **FP**(3) has exactly 12 irreducible components, each of them is rigid in the sense that it is the closure of the orbit under the action of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ of a foliation on $\mathbb{P}^2_{\mathbb{C}}$. Theorem 4.4 shows that in any even degree d, the algebraic set **FP**(d) always contains nonrigid irreducible components.

ACKNOWLEDGMENTS

This work has been partially funded by the Ministry of Science, Innovation and Universities of Spain through the grants PGC2018-095998-B-I00 and PID2021-125625NB-I00, by the Agency for Management of University and Research Grants of Catalonia through the grants 2017SGR1725 and 2021SGR01015, and by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M).

ORCID

David Marín https://orcid.org/0000-0003-4422-6418

REFERENCES

[1] A. Beltrán, M. Falla Luza, and D. Marín. *Flat 3-webs of degree one on the projective plane*, Ann. Fac. Sci. Toulouse Math. (6) **23** (2014) no. 4, 779–796.

[2] A. Beltrán, M. Falla Luza, D. Marín, and M. Nicolau, *Foliations and webs inducing Galois coverings*, Int. Math. Res. Not. IMRN **12** (2016), 3768–3827.

[3] S. Bedrouni and D. Marín, *Tissus plats et feuilletages homogènes sur le plan projectif complexe*, Bull. Soc. Math. France **146** (2018), no. 3, 479–516.

[4] S. Bedrouni and D. Marín, *Classification of foliations of degree three on $\mathbb{P}^2_{\mathbb{C}}$ with a flat Legendre transform*. Ann. Inst. Fourier (Grenoble) **71** (2021), no. 4, 1757–1790.

[5] A. Hénaut, *Planar web geometry through abelian relations and singularities*, Inspired by S. S. Chern, Nankai Tracts Math. **11** (2006), 269–295.

[6] F. Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree*, Dover Publications, New York, 2003.

[7] D. Marín and J. V. Pereira, *Rigid flat webs on the projective plane*, Asian J. Math. **17** (2013), no. 1, 163–191.

[8] J. V. Pereira and L. Pirio, *Classification of exceptional CDQL webs on compact complex surfaces*, Int. Math. Res. Not. IMRN **12** (2010), 2169–2282.

[9] J. V. Pereira and L. Pirio, *An invitation to web geometry*, vol. 2, IMPA Monographs, Springer, Cham, 2015.

[10] O. Ripoll, *Géométrie des tissus du plan et équations différentielles*, Thése de Doctorat de l'Université Bordeaux 1, 2005. http://tel.archives-ouvertes.fr/tel-00011928.

How to cite this article: S. Bedrouni and D. Marín, *A criterion for the holomorphy of the curvature of smooth planar webs and applications to dual webs of homogeneous foliations on $\mathbb{P}^2_{\mathbb{C}}$*, Math. Nachr. **297** (2024), 3964–3981. https://doi.org/10.1002/mana.202400150