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ORIGINAL PAPER

Pre-foliations of co-degree one on $\mathbb{P}^2_{\mathbb{C}}$ with a flat Legendre transform

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Abstract
A holomorphic pre-foliation $\mathscr{F} = \ell \boxtimes \mathcal{F}$ of co-degree 1 and degree d on $\mathbb{P}^2_{\mathbb{C}}$ is the data of a line ℓ of $\mathbb{P}^2_{\mathbb{C}}$ and a holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ of degree $d-1$. We study pre-foliations of co-degree 1 on $\mathbb{P}^2_{\mathbb{C}}$ with a flat Legendre transform (dual web). After having established some general results on the flatness of the dual d-web of a homogeneous pre-foliation of co-degree 1 and degree d, we describe some explicit examples and we show that up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$ there are two families and six examples of homogeneous pre-foliations of co-degree 1 and degree 3 on $\mathbb{P}^2_{\mathbb{C}}$ with a flat dual web. This allows us to prove an analogue for pre-foliations of co-degree 1 and degree 3 of a result, obtained in collaboration with D. Marín, on foliations of degree 3 with non-degenerate singularities and a flat Legendre transform. We also show that the dual web of a reduced convex pre-foliation of co-degree 1 on $\mathbb{P}^2_{\mathbb{C}}$ is flat. This is an analogue of a result on foliations of $\mathbb{P}^2_{\mathbb{C}}$ due to D. Marín and J. V. Pereira.

Keywords Pre-foliation · Homogeneous pre-foliation · Web · Flatness · Legendre transformation

Mathematics Subject Classification 14C21 · 32S65 · 53A60

Abbreviations

$\mathrm{Leg}\mathscr{F}$	LEGENDRE transform of the pre-foliation $\mathscr{F}$
$\Delta(\mathcal{W})$	Discriminant of the web $\mathcal{W}$
$\mathcal{T}_{\mathcal{H}}$	Type of the homogeneous foliation $\mathcal{H}$
$\eta(\mathcal{W})$	Fundamental form of the web $\mathcal{W}$
$K(\mathcal{W})$	Curvature of the web $\mathcal{W}$
$\mathcal{G}_{\mathcal{F}}$	GAUSS map associated to the foliation $\mathcal{F}$
$\mathrm{Sing}\mathcal{F}$	Singular locus of the foliation $\mathcal{F}$
$\mathrm{I}^{\mathrm{tr}}_{\mathcal{F}}$	Transverse part of the inflection divisor $\mathrm{I}_{\mathcal{F}}$
$\mathrm{rk}(\mathcal{W})$	Rank of the web $\mathcal{W}$
$S_{\mathcal{W}}$	Characteristic surface of the web $\mathcal{W}$
$\mathrm{C}_{\mathcal{H}}$	Tangent cone at the origin of the homogeneous foliation $\mathcal{H}$

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$\mathrm{I}_{\mathcal{F}}$	Inflection divisor of the foliation $\mathcal{F}$
$\mathrm{I}_{\mathcal{F}}^{\mathrm{inv}}$	Invariant part of the inflection divisor $\mathrm{I}_{\mathcal{F}}$
ω_0^{d-1}	$(d-2)y^{d-1}\mathrm{d}x + x\left(x^{d-2}-(d-1)y^{d-2}\right)\mathrm{d}y$
ω_4^{d-1}	$y(\sigma_d\, x^{d-2} - y^{d-2})\mathrm{d}x + x(\sigma_d\, y^{d-2} - x^{d-2})\mathrm{d}y$
ω_1^{d-1}	$y^{d-1}\mathrm{d}x - x^{d-1}\mathrm{d}y$
$\omega_2^{d-1}(\lambda,\mu)$	$(x^{d-1}+\lambda\, y^{d-1})\mathrm{d}x + (\mu\, x^{d-1} - y^{d-1})\mathrm{d}y$
$\omega_3^{d-1}(\lambda)$	$(x^{d-1}+\lambda y^{d-1})\mathrm{d}x + x^{d-1}\mathrm{d}y$
$\mathrm{CS}(\mathcal{F},\mathcal{C},s)$	CAMACHO- SAD index of the foliation $\mathcal{F}$ at the point s along the curve $\mathcal{C}$
$\tau(\mathcal{F},s)$	Tangency order of the foliation $\mathcal{F}$ with a generic line passing through the point s
$\mathcal{O}(\mathscr{F})$	Orbit of the pre-foliation $\mathscr{F}$ under the action of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$
$\mu(\mathcal{F},s)$	MILNOR number of the foliation $\mathcal{F}$ at the singular point s
$\mathrm{BB}(\mathcal{F},s)$	BAUM-BOTT invariant of the foliation $\mathcal{F}$ at the singular point s

Introduction

This article is a continuation of a series of joint works with D. MARÍN [4–7] on holomorphic foliations on the complex projective plane. For the definitions and notations used (web, discriminant $\Delta(\mathcal{W})$, homogeneous foliation, inflection divisor $\mathrm{I}_{\mathcal{F}}$, radial singularity, etc.) we refer to [4, Sections 1 and 2].

We begin by introducing the following definition, where the terminology «pre-foliation » is taken from [9].

Definition A Let $0 \leq k \leq d$ be integers. A *holomorphic pre-foliation* $\mathscr{F}$ *on* $\mathbb{P}^2_{\mathbb{C}}$ *of co-degree* k *and degree* d, or simply *of type* (k,d), is the data of a reduced complex projective curve $\mathcal{C} \subset \mathbb{P}^2_{\mathbb{C}}$ of degree k and a holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ of degree $d-k$. We write $\mathscr{F} = \mathcal{C}\boxtimes\mathcal{F}$ and call $\mathcal{C}$ (resp. $\mathcal{F}$) the *associated curve* (resp. the *associated foliation*) to $\mathscr{F}$.

Such a pre-foliation is given in homogeneous coordinates $[x:y:z] \in \mathbb{P}^2_{\mathbb{C}}$ by a 1-form of type $\Omega = F(x,y,z)\Omega_0$, where $\mathbb{C}[x,y,z]_k \ni F(x,y,z) = 0$ is a homogeneous equation of the curve $\mathcal{C}$ and Ω_0 is a homogeneous 1-form of degree $d-k+1$ defining the foliation $\mathcal{F}$, *i.e.*

$$\Omega_0 = a(x,y,z)\mathrm{d}x + b(x,y,z)\mathrm{d}y + c(x,y,z)\mathrm{d}z,$$

where a, b and c are homogeneous polynomials of degree $d-k+1$ without common factor and satisfying the EULER condition $i_{\mathrm{R}}\omega = 0$, where $\mathrm{R} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ denotes the radial vector field and i_{R} is the interior product by R.

We will denote the set of pre-foliations of type (k,d) on $\mathbb{P}^2_{\mathbb{C}}$ by $\mathbf{F}(k,d)$. It can be naturally identified with a ZARISKI open subset of the space $\mathbb{P}^{N_1}_{\mathbb{C}} \times \mathbb{P}^{N_2}_{\mathbb{C}}$, where $N_1 = \frac{k(k+3)}{2}$ and $N_2 = (d-k+2)^2 - 2$. The set $\mathbf{F}(0,d)$ describes precisely the set of foliations of degree d on $\mathbb{P}^2_{\mathbb{C}}$.

By [13], to every pre-foliation $\mathscr{F} = \mathcal{C}\boxtimes\mathcal{F}$ of degree $d \geq 1$ and co-degree $k < d$ on $\mathbb{P}^2_{\mathbb{C}}$ we can associate a d-web of degree 1 on the dual projective plane $\check{\mathbb{P}}^2_{\mathbb{C}}$, called LEGENDRE *transform* (or *dual web*) of $\mathscr{F}$ and denoted by $\mathrm{Leg}\mathscr{F}$; if $\mathscr{F}$ is given in an affine chart (x,y) of $\mathbb{P}^2_{\mathbb{C}}$ by a 1-form $\omega = f(x,y)\,(A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y)$ then, in the affine chart (p,q) of $\check{\mathbb{P}}^2_{\mathbb{C}}$ associated to the line $\{y = px - q\} \subset \mathbb{P}^2_{\mathbb{C}}$, $\mathrm{Leg}\mathscr{F}$ is described by the implicit differential equation

$$F(p,q,x) := f(x, px-q)\,(A(x, px-q) + pB(x, px-q)) = 0, \qquad \text{with} \qquad x = \frac{\mathrm{d}q}{\mathrm{d}p}.$$

When $k \geq 1$, $\mathrm{Leg}\mathscr{F}$ decomposes as $\mathrm{Leg}\mathscr{F} = \mathrm{Leg}\mathcal{C} \boxtimes \mathrm{Leg}\mathcal{F}$, where $\mathrm{Leg}\mathcal{C}$ is the algebraic k-web on $\check{\mathbb{P}}^2_{\mathbb{C}}$ defined by the equation $f(x, px-q) = 0$ and $\mathrm{Leg}\mathcal{F}$ is the irreducible $(d-k)$-web of degree 1 on $\check{\mathbb{P}}^2_{\mathbb{C}}$ given by $A(x, px-q) + pB(x, px-q) = 0$.

Conversely, every decomposable d-web of degree 1 on $\check{\mathbb{P}}^2_{\mathbb{C}}$ is necessarily the Legendre transform of a certain pre-foliation on $\mathbb{P}^2_{\mathbb{C}}$ of type (k, d), with $1 \leq k < d$.

The curvature of a web $\mathcal{W}$ on $\mathbb{P}^2_{\mathbb{C}}$ is a meromorphic 2-form with poles along the discriminant $\Delta(\mathcal{W})$. A web with zero curvature is called flat. The flatness of a web $\mathcal{W}$ on $\mathbb{P}^2_{\mathbb{C}}$ is characterized by the holomorphy of its curvature $K(\mathcal{W})$ along the generic points of $\Delta(\mathcal{W})$, *see* §1.

The subset $\mathbf{FP}(k, d)$ of $\mathbf{F}(k, d)$ consisting of $\mathscr{F} \in \mathbf{F}(k, d)$ such that $\mathrm{Leg}\mathscr{F}$ is flat is Zariski closed in $\mathbf{F}(k, d)$.

Problem 1 *Let d and k be integers such that $d \geq 3$ and $0 \leq k < d$. Describe certain irreducible components of* $\mathbf{FP}(k, d)$.

This problem is inspired by a part of Problem 9.3 of [13] which consists in the description of certain irreducible components of the space of flat d-webs of degree 1 on $\check{\mathbb{P}}^2_{\mathbb{C}}$. The first case $(k, d) = (0, 3)$ has been completely treated in [6]. In fact, the author and Marín (*see* [4, Sections 3–6]) first studied the flatness of dual webs of homogeneous foliations of $\mathbb{P}^2_{\mathbb{C}}$, and they showed that it is possible to reduce the study of the flatness of dual webs of certain inhomogeneous foliations to the homogeneous framework. This work ([4, Theorems 5.1 and 6.1]) then allowed to show that $\mathbf{FP}(0, 3)$ has exactly 12 irreducible components, *see* [6, Theorem D]. In what follows we are interested in pre-foliations of co-degree 1, *i.e.* whose associated curve is a line. We will not look for irreducible components of $\mathbf{FP}(1, d)$, but will adapt the approach of [4] to co-degree one pre-foliations.

Definition B A pre-foliation on $\mathbb{P}^2_{\mathbb{C}}$ is said to be *homogeneous* if there is an affine chart (x, y) of $\mathbb{P}^2_{\mathbb{C}}$ in which it is invariant under the action of the group of homotheties $(x, y) \longmapsto \lambda(x, y),\ \lambda \in \mathbb{C}^*$.

A homogeneous pre-foliation $\mathscr{H}$ of type $(1, d)$ on $\mathbb{P}^2_{\mathbb{C}}$ is then of the form $\mathscr{H} = \ell \boxtimes \mathcal{H}$, where $\mathcal{H}$ is a homogeneous foliation of degree $d-1$ on $\mathbb{P}^2_{\mathbb{C}}$ and where ℓ is a line passing through the origin O or $\ell = L_\infty$.

Theorem 3.1 of [4] states that the web $\mathrm{Leg}\mathcal{H}$ is flat if and only if its curvature is holomorphic on the transverse part of its discriminant $\Delta(\mathrm{Leg}\mathcal{H})$. We prove in Section §3 a similar result (Theorem 3.7) for the web $\mathrm{Leg}\mathscr{H}$.

When ℓ passes through the origin, we establish effective criteria for the holomorphy of the curvature of $\mathrm{Leg}\mathscr{H}$ on certain irreducible components of the discriminant $\Delta(\mathrm{Leg}\mathscr{H})$ (Theorems 3.13 and 3.18). In fact, Theorems 3.7, 3.13 and 3.18 provide a complete characterization of the flatness of $\mathrm{Leg}\mathscr{H}$.

When $\ell = L_\infty$ we show (Theorem 3.1) that the webs $\mathrm{Leg}\mathcal{H}$ and $\mathrm{Leg}\mathscr{H}$ have the same curvature; in particular the flatness of $\mathrm{Leg}\mathscr{H}$ is equivalent to that of $\mathrm{Leg}\mathcal{H}$. More particularly, in degree $d = 3$ the web $\mathrm{Leg}\mathscr{H}$ is flat (Corollary 3.2).

Recall (*see* [13]) that a holomorphic foliation on $\mathbb{P}^2_{\mathbb{C}}$ is said to be *convex* if its leaves other than straight lines have no inflection points. Note (*see* [14]) that if $\mathcal{F}$ is a foliation of degree $d \geq 1$ on $\mathbb{P}^2_{\mathbb{C}}$, then $\mathcal{F}$ cannot have more than $3d$ (distinct) invariant lines. Moreover, if this bound is reached, then $\mathcal{F}$ is necessarily convex; in this case $\mathcal{F}$ is said to be *reduced convex*. We naturally extend the notions of convexity and reduced convexity of foliations to pre-foliations by putting:

Definition C Let $\mathscr{F} = \mathcal{C} \boxtimes \mathcal{F}$ be a pre-foliation on $\mathbb{P}^2_{\mathbb{C}}$. We say that $\mathscr{F}$ is *convex* (resp. *reduced convex*) if the foliation $\mathcal{F}$ is convex (resp. reduced convex) and if moreover the curve $\mathcal{C}$ is invariant by $\mathcal{F}$.

From this definition and Theorem 3.7 we will deduce the following corollary, which is an analogue of Corollary 3.4 of [4].

Corollary D *(Corollary 3.11) The dual web of a homogeneous convex pre-foliation of co-degree* 1 *on* $\mathbb{P}^2_{\mathbb{C}}$ *is flat.*

In §4 we give an application of the results of §3 to homogeneous pre-foliations $\mathscr{H} = \ell \boxtimes \mathcal{H}$ of co-degree 1 such that the degree of type of $\mathcal{H}$ is equal to 2, *i.e.* $\deg \mathcal{T}_{\mathcal{H}} = 2$ (*see* [4, Definition 2.3] for the definitions of the type $\mathcal{T}_{\mathcal{H}}$ and the degree of type $\deg \mathcal{T}_{\mathcal{H}}$). More precisely, we describe, up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$, all homogeneous pre-foliations $\mathscr{H} = \ell \boxtimes \mathcal{H}$ of co-degree 1 and degree $d \geq 3$ such that $\deg \mathcal{T}_{\mathcal{H}} = 2$ and the d-web $\mathrm{Leg}\mathscr{H}$ is flat (Proposition 4.4). We obtain in particular, for $d = 3$, the classification up to automorphism of homogeneous pre-foliations of type $(1, 3)$ on $\mathbb{P}^2_{\mathbb{C}}$ whose dual 3-web is flat: *up to automorphism of* $\mathbb{P}^2_{\mathbb{C}}$, *there are two families and six examples of homogeneous pre-foliations of co-degree* 1 *and degree* 3 *on* $\mathbb{P}^2_{\mathbb{C}}$ *with a flat* LEGENDRE *transform*, *see* Corollary 4.5.

In 2013 MARÍN and PEREIRA [13, Theorem 4.2] proved that the dual web of a reduced convex foliation on $\mathbb{P}^2_{\mathbb{C}}$ is flat. We show in §5 the following analogous result for co-degree one pre-foliations.

Theorem E *Let* $\mathscr{F} = \ell \boxtimes \mathcal{F}$ *be a reduced convex pre-foliation of co-degree* 1 *and degree* $d \geq 3$ *on* $\mathbb{P}^2_{\mathbb{C}}$. *Then the* d*-web* $\mathrm{Leg}\mathscr{F}$ *is flat.*

The following problem then arises.

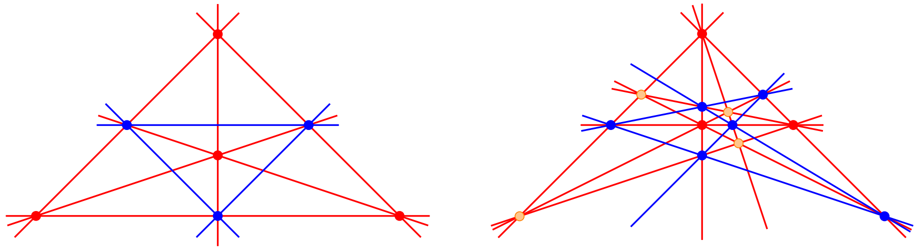
Problem 2 *Let* $\mathcal{F}$ *be a reduced convex foliation of degree greater than or equal to* 2 *on* $\mathbb{P}^2_{\mathbb{C}}$ *and let* ℓ *be a line of* $\mathbb{P}^2_{\mathbb{C}}$ *which is not invariant by* $\mathcal{F}$. *Determine the relative position of the line* ℓ *with respect to the invariant lines of* $\mathcal{F}$ *such that the dual web of the pre-foliation* $\ell \boxtimes \mathcal{F}$ *is flat.*

To our knowledge the only reduced convex foliations known in the literature are those presented in [13, Table 1.1]: the FERMAT foliation $\mathcal{F}_0^{d-1}$ of degree $d-1$, the HESSE foliation $\mathcal{F}_H^4$ of degree 4, the HILBERT modular foliation $\mathcal{F}_H^5$ of degree 5 and the HESSE foliation $\mathcal{F}_H^7$ of degree 7 defined in affine chart respectively by the 1-forms

$$\overline{\omega}_0^{d-1} = x\mathrm{d}y - y\mathrm{d}x + y^{d-1}\mathrm{d}x - x^{d-1}\mathrm{d}y,$$
$$\omega_H^4 = y(2x^3 - y^3 - 1)\mathrm{d}x + x(2y^3 - x^3 - 1)\mathrm{d}y,$$
$$\omega_H^5 = (y^2-1)(y^2-(\sqrt{5}-2)^2)(y+\sqrt{5}x)\mathrm{d}x - (x^2-1)(x^2-(\sqrt{5}-2)^2)(x+\sqrt{5}y)\mathrm{d}y,$$
$$\omega_H^7 = (y^3-1)(y^3+7x^3+1)y\mathrm{d}x - (x^3-1)(x^3+7y^3+1)x\mathrm{d}y.$$

The following two propositions, which will be proved in §5, give an answer to Problem 2 in the case of the FERMAT foliation $\mathcal{F}_0^{d-1}$ and the HESSE foliation $\mathcal{F}_H^4$.

Proposition F *Let $d \geq 3$ be an integer and let ℓ be a line of $\mathbb{P}^2_{\mathbb{C}}$. Assume that ℓ is not invariant by the* FERMAT *foliation $\mathcal{F}_0^{d-1}$ and that the d-web $\mathrm{Leg}(\ell \boxtimes \mathcal{F}_0^{d-1})$ is flat. Then $d \in \{3, 4\}$ and the line ℓ joins two (resp. three) singularities (necessarily non-radial) of $\mathcal{F}_0^{d-1}$ if $d = 3$ (resp. if $d = 4$).*



$d = 3$ $d = 4$

Relative positions of the line ℓ (in blue) with respect to the invariant lines (in red) of the FERMAT foliations $\mathcal{F}_0^2$ and $\mathcal{F}_0^3$. The foliation $\mathcal{F}_0^2$ ($d = 3$) has 4 radial singularities (red points) and 3 non-radial singularities (blue points) with BAUM-BOTT invariant 0. The foliation $\mathcal{F}_0^3$ ($d = 4$) admits 7 radial singularities, 4 of order one (orange points) and 3 of order two (red points), and 6 non-radial singularities with BAUM-BOTT invariant $-\frac{1}{2}$.

Proposition G *Let ℓ be a line of $\mathbb{P}^2_{\mathbb{C}}$ which is not invariant by the* HESSE *foliation $\mathcal{F}_H^4$. Assume that the 5-web $\mathrm{Leg}(\ell \boxtimes \mathcal{F}_H^4)$ is flat. Then the line ℓ passes through four (necessarily non-radial) singularities of $\mathcal{F}_H^4$.*

The idea of the proofs of Propositions F and G will be to reduce to the homogeneous case, by showing that the closures of the $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$-orbits of the pre-foliations $\ell \boxtimes \mathcal{F}_0^{d-1}$ and $\ell \boxtimes \mathcal{F}_H^4$ contain homogeneous pre-foliations.

Theorem 6.1 of [4] says that every foliation of degree 3 on $\mathbb{P}^2_{\mathbb{C}}$ with non-degenerate singularities and a flat LEGENDRE transform is linearly conjugate to the FERMAT foliation $\mathcal{F}_0^3$. We prove in §6 the following similar result for pre-foliations of co-degree 1 and degree 3.

Theorem H *Let $\mathscr{F} = \ell \boxtimes \mathcal{F}$ be a pre-foliation of co-degree 1 and degree 3 on $\mathbb{P}^2_{\mathbb{C}}$. Assume that the foliation $\mathcal{F}$ has only non-degenerate singularities and that the 3-web $\mathrm{Leg}\mathscr{F}$ is flat. Then $\mathcal{F}$ is linearly conjugate to the* FERMAT *foliation $\mathcal{F}_0^2$, and the line ℓ is either invariant by $\mathcal{F}$ or it joins two non-radial singularities of $\mathcal{F}$.*

The proof of this theorem will essentially use the classification of homogeneous pre-foliations of type $(1, 3)$ on $\mathbb{P}^2_{\mathbb{C}}$ whose dual web is flat (Corollary 4.5).

1 Reminders on the fundamental form and curvature of a web

In this section, we briefly recall the definitions of the fundamental form and the curvature of a d-web $\mathcal{W}$. Let us first assume that $\mathcal{W}$ is a germ of completely decomposable d-web on $(\mathbb{C}^2, 0)$, $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_d$. For $i = 1, \ldots, d$, let ω_i be a 1-form with an isolated singularity at 0 defining the foliation $\mathcal{F}_i$. Following [15], for each triple (r, s, t) with $1 \leq r < s < t \leq d$, one defines $\eta_{rst} = \eta(\mathcal{F}_r \boxtimes \mathcal{F}_s \boxtimes \mathcal{F}_t)$ as the unique meromorphic 1-form satisfying the following equalities:

$$\begin{cases} \mathrm{d}(\delta_{st}\,\omega_r) = \eta_{rst} \wedge \delta_{st}\,\omega_r \\ \mathrm{d}(\delta_{tr}\,\omega_s) = \eta_{rst} \wedge \delta_{tr}\,\omega_s \\ \mathrm{d}(\delta_{rs}\,\omega_t) = \eta_{rst} \wedge \delta_{rs}\,\omega_t \end{cases} \tag{1.1}$$

where δ_{ij} denotes the function defined by $\omega_i \wedge \omega_j = \delta_{ij}\,\mathrm{d}x \wedge \mathrm{d}y$. One calls *fundamental form* of the web $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_d$ the 1-form

$$\eta(\mathcal{W}) = \eta(\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_d) = \sum_{1 \le r < s < t \le d} \eta_{rst}. \tag{1.2}$$

One can easily check that $\eta(\mathcal{W})$ is a meromorphic 1-form with poles along the discriminant $\Delta(\mathcal{W})$ of $\mathcal{W}$, and that it is well-defined up to addition of a closed logarithmic 1-form $\dfrac{\mathrm{d}f}{f}$ with $f \in \mathcal{O}^*(\mathbb{C}^2, 0)$ (*cf.* [4, 17]).

Now, if $\mathcal{W}$ is a (not necessarily completely decomposable) d-web on a complex surface M then its pull-back by a suitable ramified GALOIS covering is completely decomposable. The invariance of the fundamental form of this new web by the action of the GALOIS group allows us to descend it to a global meromorphic 1-form $\eta(\mathcal{W})$ on M, with poles along the discriminant of $\mathcal{W}$ (*see* [13]).

The *curvature* of the web $\mathcal{W}$ is by definition the 2-form

$$K(\mathcal{W}) = \mathrm{d}\,\eta(\mathcal{W}).$$

It is a meromorphic 2-form with poles along the discriminant $\Delta(\mathcal{W})$ of $\mathcal{W}$, canonically associated to $\mathcal{W}$. More precisely, for any dominant holomorphic map φ, one has $K(\varphi^*\mathcal{W}) = \varphi^* K(\mathcal{W})$.

A d-web $\mathcal{W}$ is said to be *flat* if its curvature $K(\mathcal{W})$ vanishes identically.

Let us finally note that a d-web $\mathcal{W}$ on $\mathbb{P}^2_{\mathbb{C}}$ is flat if and only if its curvature is holomorphic over the generic points of the irreducible components of $\Delta(\mathcal{W})$. This follows from the holomorphy of $K(\mathcal{W})$ on $\mathbb{P}^2_{\mathbb{C}} \setminus \Delta(\mathcal{W})$ and from the fact that there are no holomorphic 2-forms on $\mathbb{P}^2_{\mathbb{C}}$ other than the zero 2-form.

2 Discriminant of the dual web of a co-degree one pre-foliation

Recall that if $\mathcal{F}$ is a foliation on $\mathbb{P}^2_{\mathbb{C}}$, the GAUSS map of $\mathcal{F}$ is the rational map $\mathcal{G}_{\mathcal{F}} : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \check{\mathbb{P}}^2_{\mathbb{C}}$ defined at every regular point m of $\mathcal{F}$ by $\mathcal{G}_{\mathcal{F}}(m) = \mathrm{T}^{\mathbb{P}}_m\mathcal{F}$, where $\mathrm{T}^{\mathbb{P}}_m\mathcal{F}$ denotes the tangent line to the leaf of $\mathcal{F}$ passing through m. If $\mathcal{C}$ is a curve on $\mathbb{P}^2_{\mathbb{C}}$ passing through some singular points of $\mathcal{F}$, one defines $\mathcal{G}_{\mathcal{F}}(\mathcal{C})$ as the closure of $\mathcal{G}_{\mathcal{F}}(\mathcal{C} \setminus \mathrm{Sing}\mathcal{F})$.

Lemma 2.1 *Let $\mathscr{F} = \ell \boxtimes \mathcal{F}$ be a pre-foliation of co-degree 1 on $\mathbb{P}^2_{\mathbb{C}}$.*
1. *If the line ℓ is invariant by $\mathcal{F}$, then*

$$\Delta(\mathrm{Leg}\mathscr{F}) = \Delta(\mathrm{Leg}\mathcal{F}) \cup \check{\Sigma}^{\ell}_{\mathcal{F}},$$

where $\check{\Sigma}^{\ell}_{\mathcal{F}}$ denotes the set of lines dual to the points of $\Sigma^{\ell}_{\mathcal{F}} := \mathrm{Sing}\mathcal{F} \cap \ell$.
2. *If the line ℓ is not invariant by $\mathcal{F}$, then*

$$\Delta(\mathrm{Leg}\mathscr{F}) = \Delta(\mathrm{Leg}\mathcal{F}) \cup \mathcal{G}_{\mathcal{F}}(\ell) \cup \check{\Sigma}^{\ell}_{\mathcal{F}}.$$

Proof We have

$$\Delta(\mathrm{Leg}\mathscr{F}) = \Delta(\mathrm{Leg}\mathcal{F}) \cup \mathrm{Tang}(\mathrm{Leg}\ell, \mathrm{Leg}\mathcal{F}).$$

When ℓ is not invariant by $\mathcal{F}$, we obtain by an argument of [1, page 33] that

$$\mathrm{Tang}(\mathrm{Leg}\ell, \mathrm{Leg}\mathcal{F}) = \mathcal{G}_{\mathcal{F}}(\ell) \cup \check{\Sigma}_{\mathcal{F}}^{\ell}.$$

Let us assume that ℓ is invariant by $\mathcal{F}$ and show that $\mathrm{Tang}(\mathrm{Leg}\ell, \mathrm{Leg}\mathcal{F}) = \check{\Sigma}_{\mathcal{F}}^{\ell}$. Let $s \in \Sigma_{\mathcal{F}}^{\ell}$. The fact that $s \in \ell$ (resp. $s \in \mathrm{Sing}\mathcal{F}$) implies that the line $\check{s}$ dual to s is invariant by $\mathrm{Leg}\ell$ (resp. by $\mathrm{Leg}\mathcal{F}$). Thus $\check{s} \subset \mathrm{Tang}(\mathrm{Leg}\ell, \mathrm{Leg}\mathcal{F})$, hence $\check{\Sigma}_{\mathcal{F}}^{\ell} \subset \mathrm{Tang}(\mathrm{Leg}\ell, \mathrm{Leg}\mathcal{F})$. Conversely, let $\mathcal{C}$ be an irreducible component of $\mathrm{Tang}(\mathrm{Leg}\ell, \mathrm{Leg}\mathcal{F})$. Let us show that $\mathcal{C}$ is invariant by $\mathrm{Leg}\mathcal{F}$. Assume by contradiction that $\mathcal{C}$ is transverse to $\mathrm{Leg}\mathcal{F}$. Let m be a generic point of $\mathcal{C}$. Denote by $\check{\ell} \in \check{\mathbb{P}}_{\mathbb{C}}^{2}$ the dual point of ℓ; then the line $(\check{\ell}m)$ is not invariant by $\mathrm{Leg}\mathcal{F}$ and is tangent to $\mathrm{Leg}\mathcal{F}$ at m. Since ℓ is $\mathcal{F}$-invariant, the point $\check{\ell}$ is singular for $\mathrm{Leg}\mathcal{F}$; it is therefore also a tangency point between $\mathrm{Leg}\mathcal{F}$ and $(\check{\ell}m)$. The number of tangency points between $\mathrm{Leg}\mathcal{F}$ and $(\check{\ell}m)$ is then ≥ 2, which contradicts the equality $\deg(\mathrm{Leg}\mathcal{F}) = 1$. Hence the invariance of $\mathcal{C}$ by $\mathrm{Leg}\mathcal{F}$ is proved. Then $\mathcal{C}$ is also invariant by $\mathrm{Leg}\ell$ and is therefore a line passing through $\check{\ell}$. There therefore exists $s \in \mathrm{Sing}\mathcal{F}$ such that $\mathcal{C} = \check{s}$; since $\check{\ell} \in \mathcal{C}$, we have $s \in \ell$ and therefore $s \in \Sigma_{\mathcal{F}}^{\ell}$. Consequently, $\mathcal{C} \subset \check{\Sigma}_{\mathcal{F}}^{\ell}$. □

We will now apply the above lemma to the case of a homogeneous pre-foliation $\mathscr{H} = \ell \boxtimes \mathcal{H}$ of co-degree 1 on $\mathbb{P}_{\mathbb{C}}^{2}$. If $\deg \mathscr{H} = d$, the homogeneous foliation $\mathcal{H}$ is given, for a suitable choice of affine coordinates (x, y), by a homogeneous 1-form

$$\omega = A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y, \quad \text{where } A, B \in \mathbb{C}[x, y]_{d-1} \text{ with } \gcd(A, B) = 1.$$

If $\ell = L_{\infty}$ then ℓ is invariant by $\mathcal{H}$ and Lemma 2.1 ensures that

$$\Delta(\mathrm{Leg}\mathscr{H}) = \Delta(\mathrm{Leg}\mathcal{H}) \cup \check{\Sigma}_{\mathcal{H}}^{\infty},$$

where $\check{\Sigma}_{\mathcal{H}}^{\infty}$ denotes the set of lines dual to the points of $\Sigma_{\mathcal{H}}^{\infty} := \mathrm{Sing}\mathcal{H} \cap L_{\infty}$.

Assume that ℓ passes through the origin. If ℓ is not invariant by $\mathcal{H}$, then, according to [4, Proposition 2.2], we have $\Sigma_{\mathcal{H}}^{\ell} = \{O\}$. Since the line $\check{O}$ dual to O is contained in $\Delta(\mathrm{Leg}\mathcal{H})$ by [4, Lemma 3.2], it follows from Lemma 2.1 that

$$\Delta(\mathrm{Leg}\mathscr{H}) = \Delta(\mathrm{Leg}\mathcal{H}) \cup \mathcal{G}_{\mathcal{H}}(\ell).$$

If ℓ is invariant by $\mathcal{H}$, then the point $s := L_{\infty} \cap \ell$ is singular for $\mathcal{H}$ and, by [4, Proposition 2.2], we have $\Sigma_{\mathcal{H}}^{\ell} = \{O, s\}$. Denoting by $\check{s}$ the dual line of the point s, the inclusion $\check{O} \subset \Delta(\mathrm{Leg}\mathcal{H})$ and Lemma 2.1 imply that

$$\Delta(\mathrm{Leg}\mathscr{H}) = \Delta(\mathrm{Leg}\mathcal{H}) \cup \check{s}.$$

According to [4, Lemma 3.2], the discriminant of $\mathrm{Leg}\mathcal{H}$ decomposes as

$$\Delta(\mathrm{Leg}\mathcal{H}) = \mathcal{G}_{\mathcal{H}}(\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}}) \cup \check{\Sigma}_{\mathcal{H}}^{\mathrm{rad}} \cup \check{O},$$

where $\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}}$ denotes the transverse inflection divisor of $\mathcal{H}$ and $\check{\Sigma}_{\mathcal{H}}^{\mathrm{rad}}$ is the set of lines dual to the radial singularities of $\mathcal{H}$ (*see* [4, §1.3] for precise definitions of these notions). Recall however that to the homogeneous foliation $\mathcal{H}$ one can also associate the rational map $\underline{\mathcal{G}}_{\mathcal{H}} : \mathbb{P}_{\mathbb{C}}^{1} \to \mathbb{P}_{\mathbb{C}}^{1}$ defined by

$$\underline{\mathcal{G}}_{\mathcal{H}}([y : x]) = [-A(x, y) : B(x, y)],$$

and that this map allows us to completely determine the divisor $\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}}$ and the set $\Sigma_{\mathcal{H}}^{\mathrm{rad}}$ (*see* [4, Section 2]):

- $\Sigma_{\mathcal{H}}^{\mathrm{rad}}$ consists of $[b:a:0]\in L_\infty$ such that $[a:b]\in\mathbb{P}^1_{\mathbb{C}}$ is a fixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$;
- $\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}}=\prod\limits_i T_i^{n_i}$, where $T_i=(b_i y-a_i x=0)$ and $[a_i:b_i]\in\mathbb{P}^1_{\mathbb{C}}$ is a non-fixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$ of multiplicity n_i.

From the above considerations, we deduce the following lemma.

Lemma 2.2 *Let $\mathscr{H}=\ell\boxtimes\mathcal{H}$ be a homogeneous pre-foliation of co-degree 1 on $\mathbb{P}^2_{\mathbb{C}}$.*
1. If $\ell=L_\infty$ then

$$\Delta(\mathrm{Leg}\mathscr{H})=\Delta(\mathrm{Leg}\mathcal{H})\cup\check{\Sigma}_{\mathcal{H}}^{\infty}=\mathcal{G}_{\mathcal{H}}(\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}})\cup\check{\Sigma}_{\mathcal{H}}^{\infty}\cup\check{O}.$$

2. If the line ℓ passes through the origin, then

$$\Delta(\mathrm{Leg}\mathscr{H})=\Delta(\mathrm{Leg}\mathcal{H})\cup D_\ell=\mathcal{G}_{\mathcal{H}}(\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}})\cup\check{\Sigma}_{\mathcal{H}}^{\mathrm{rad}}\cup\check{O}\cup D_\ell,$$

where the component D_ℓ is defined as follows. If ℓ is invariant by $\mathcal{H}$, then $D_\ell:=\check{s}$ is the dual line of the point $s=L_\infty\cap\ell\in\mathrm{Sing}\mathcal{H}$. If ℓ is not invariant by $\mathcal{H}$, then $D_\ell:=\mathcal{G}_{\mathcal{H}}(\ell)$.

3 Flatness of the dual web of a co-degree one homogeneous pre-foliation

Our first result shows that, for a homogeneous foliation $\mathcal{H}$ on $\mathbb{P}^2_{\mathbb{C}}$, the webs $\mathrm{Leg}\mathcal{H}$ and $\mathrm{Leg}(L_\infty\boxtimes\mathcal{H})$ have the same curvature, so that we have equivalence between the flatness of $\mathrm{Leg}\mathcal{H}$ and that of $\mathrm{Leg}(L_\infty\boxtimes\mathcal{H})$.

Theorem 3.1 *Let $d\geq 3$ be an integer and let $\mathcal{H}$ be a homogeneous foliation of degree $d-1$ on $\mathbb{P}^2_{\mathbb{C}}$. Then*

$$K(\mathrm{Leg}(L_\infty\boxtimes\mathcal{H}))=K(\mathrm{Leg}\mathcal{H}).$$

In particular, the d-web $\mathrm{Leg}(L_\infty\boxtimes\mathcal{H})$ is flat if and only if the $(d-1)$-web $\mathrm{Leg}\mathcal{H}$ is flat.

Corollary 3.2 *Let $\mathcal{H}$ be a homogeneous foliation of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$. Then the 3-web $\mathrm{Leg}(L_\infty\boxtimes\mathcal{H})$ is flat.*

To establish Theorem 3.1, we will need the following definition and theorem.

Definition 3.3 ([12]) Let $\mathcal{W}=\mathcal{F}_1\boxtimes\cdots\boxtimes\mathcal{F}_d$ be a regular d-web on $(\mathbb{C}^2,0)$. A *transverse symmetry* of $\mathcal{W}$ is a germ of vector field X which is transverse to the foliations $\mathcal{F}_i$ $(i=1,\ldots,d)$ and whose local flow $\exp(t\mathrm{X})$ preserves the $\mathcal{F}_i$'s.

Theorem 3.4 *Let $d\geq 3$ be an integer and let $\mathcal{W}_{d-1}$ be a regular $(d-1)$-web on $(\mathbb{C}^2,0)$ which admits a transverse symmetry X. Denote by $\mathcal{F}_{\mathrm{X}}$ the foliation defined by X. Then*

$$K(\mathcal{F}_{\mathrm{X}}\boxtimes\mathcal{W}_{d-1})=K(\mathcal{W}_{d-1}).$$

In particular, the d-web $\mathcal{F}_{\mathrm{X}}\boxtimes\mathcal{W}_{d-1}$ is flat if and only if the $(d-1)$-web $\mathcal{W}_{d-1}$ is flat.

Before proving this theorem, let us briefly recall the definition of the rank $\mathrm{rk}(\mathcal{W})$ of a regular d-web $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_d$ on $(\mathbb{C}^2, 0)$. For $1 \leq i \leq d$, let ω_i be a 1-form defining the foliation $\mathcal{F}_i$. One defines the $\mathbb{C}$-vector space $\mathcal{A}(\mathcal{W})$ of *abelian relations* of $\mathcal{W}$ by

$$\mathcal{A}(\mathcal{W}) := \Big\{(\eta_1, \ldots, \eta_d) \in (\Omega^1(\mathbb{C}^2, 0))^d \;\Big|\; \forall i = 1, \ldots, d,\ \mathrm{d}\eta_i = 0,\ \eta_i \wedge \omega_i = 0 \ \text{ and } \sum_{i=1}^{d} \eta_i = 0\Big\}.$$

Then $\mathrm{rk}(\mathcal{W}) := \dim_{\mathbb{C}} \mathcal{A}(\mathcal{W})$. One has the following optimal bound (*cf.* [16, Chapter 2]):

$$\mathrm{rk}(\mathcal{W}) \leq \pi_d := \frac{(d-1)(d-2)}{2}.$$

Recall also that every d-web of maximal rank (*i.e.* of rank π_d) is necessarily flat by MIHAILEANU's criterion (*cf.* [16, Theorem 6.3.4]).

In fact, Theorem 3.4 is an analogue for flat webs of a result on webs of maximal rank, due to MARÍN-PEREIRA-PIRIO, namely:

Theorem 3.5 *([12],* Theorem 1) With the notations of Theorem 3.4, one has

$$\mathrm{rk}(\mathcal{F}_{\mathrm{X}} \boxtimes \mathcal{W}_{d-1}) = \mathrm{rk}(\mathcal{W}_{d-1}) + (d-2).$$

In particular, $\mathcal{F}_{\mathrm{X}} \boxtimes \mathcal{W}_{d-1}$ is of maximal rank if and only if $\mathcal{W}_{d-1}$ is of maximal rank.

The proof of Theorem 3.4 consists essentially in applying this result for $d = 3$.

Proof of Theorem 3.4 Writing $\mathcal{W}_{d-1} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_{d-1}$, we have

$$K(\mathcal{F}_{\mathrm{X}} \boxtimes \mathcal{W}_{d-1}) = K(\mathcal{W}_{d-1}) + \sum_{1 \leq i < j \leq d-1} K(\mathcal{W}_3^{i,j}),$$

where $\mathcal{W}_3^{i,j} := \mathcal{F}_{\mathrm{X}} \boxtimes \mathcal{F}_i \boxtimes \mathcal{F}_j$. Moreover, since X is a transverse symmetry of the 2-web $\mathcal{F}_i \boxtimes \mathcal{F}_j$ and since every 2-web is of maximal rank, equal to 0, Theorem 1 of [12] (*cf.* Theorem 3.5 above) implies that the 3-web $\mathcal{W}_3^{i,j}$ is of maximal rank, equal to 1, so that $K(\mathcal{W}_3^{i,j}) \equiv 0$, hence the announced equality holds. □

Proof of Theorem 3.1 By [4, Section 2], we can locally decompose the d-web $\mathrm{Leg}(L_\infty \boxtimes \mathcal{H})$ as

$$\mathrm{Leg}(L_\infty \boxtimes \mathcal{H}) = \mathrm{Leg}(L_\infty) \boxtimes \mathcal{W}_{d-1},$$

where $\mathcal{W}_{d-1} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_{d-1}$ and, for any $i \in \{1, \ldots, d-1\}$, $\mathcal{F}_i$ is given by $\check{\omega}_i := \lambda_i(p)\mathrm{d}q - q\mathrm{d}p$, with $\lambda_i(p) = p - p_i(p)$ and $\{p_i(p)\} = \underline{\mathcal{G}}_{\mathcal{H}}^{-1}(p)$. Now, the vector field $\mathrm{X} := q\frac{\partial}{\partial q}$ defines the radial foliation $\mathrm{Leg}(L_\infty)$ and is a transverse symmetry of the web $\mathcal{W}_{d-1}$. Therefore, $K(\mathrm{Leg}(L_\infty \boxtimes \mathcal{H})) = K(\mathrm{Leg}\mathcal{H})$ by Theorem 3.4. □

Remark 3.6 We can also prove Theorem 3.1 directly, without using results on webs of maximal rank. Indeed, putting $\mathcal{W}_3^{i,j} := \mathrm{Leg}(L_\infty) \boxtimes \mathcal{F}_i \boxtimes \mathcal{F}_j$, for all $i, j \in \{1, \ldots, d-1\}$ with $i \neq j$, we have

$$K(\mathrm{Leg}(L_\infty \boxtimes \mathcal{H})) = K(\mathrm{Leg}\mathcal{H}) + \sum_{1 \leq i < j \leq d-1} K(\mathcal{W}_3^{i,j}).$$

The foliation $\mathrm{Leg}(L_\infty)$ being defined by $\check{\omega}_0 := \mathrm{d}p$, a direct computation using formula (1.1) shows that

$$\eta(\mathcal{W}_3^{i,j}) = \frac{\mathrm{d}\Big((\lambda_i\lambda_j)(p)\Big)}{(\lambda_i\lambda_j)(p)} + \frac{\mathrm{d}q}{q},$$

so that $K(\mathcal{W}_3^{i,j}) = \mathrm{d}\eta(\mathcal{W}_3^{i,j}) \equiv 0$, hence $K(\mathrm{Leg}(L_\infty \boxtimes \mathcal{H})) = K(\mathrm{Leg}\mathcal{H})$.

The following theorem gives an important characterization of the flatness of the dual web of a co-degree one homogeneous pre-foliation.

Theorem 3.7 *Let $\mathscr{H} = \ell \boxtimes \mathcal{H}$ be a homogeneous pre-foliation of type $(1, d)$ on $\mathbb{P}^2_{\mathbb{C}}$ with $d \geq 3$. If the line ℓ is invariant (resp. not invariant) by $\mathcal{H}$, then the d-web $\mathrm{Leg}\mathscr{H}$ is flat if and only if its curvature $K(\mathrm{Leg}\mathscr{H})$ is holomorphic on $\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$ (resp. on $\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}) \cup D_\ell = \mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}} \cup \ell)$).*

To prove this theorem, we will need the following lemma, which is a reformulation in terms of homogeneous pre-foliations of Lemma 3.1 of [2] formulated in terms of homogeneous foliations not necessarily saturated.

Lemma 3.8 *([2],* Lemma 3.1) Let $\mathscr{H}$ be a homogeneous pre-foliation on $\mathbb{P}^2_{\mathbb{C}}$. If the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic on $\check{\mathbb{P}}^2_{\mathbb{C}} \setminus \check{O}$, then $\mathrm{Leg}\mathscr{H}$ is flat.

We will also need the following proposition, which has its own interest.

Proposition 3.9 *Let $\mathcal{W}_\nu$ be a germ of ν-web on $(\mathbb{C}^2, 0)$ with $\nu \geq 2$. Assume that $\Delta(\mathcal{W}_\nu)$ has an irreducible component C totally invariant by $\mathcal{W}_\nu$ and of minimal multiplicity $\nu - 1$. Let $\mathcal{F}$ be a germ of foliation on $(\mathbb{C}^2, 0)$ leaving C invariant and let $\mathcal{W}_{d-\nu-1}$ be a germ of regular $(d - \nu - 1)$-web on $(\mathbb{C}^2, 0)$ transverse to C. Then the curvature of the d-web $\mathcal{W} = \mathcal{F} \boxtimes \mathcal{W}_\nu \boxtimes \mathcal{W}_{d-\nu-1}$ is holomorphic along C.*

Proof As in the beginning of the proof of [13, Proposition 2.6], we choose a local coordinate system $(U, (x, y))$ such that $C \cap U = \{y = 0\}$, $\mathrm{T}\mathcal{F}|_U = \{\mathrm{d}y + yh(x, y)\mathrm{d}x = 0\}$,

$$\mathrm{T}\mathcal{W}_\nu|_U = \Big\{\mathrm{d}y^\nu + y\Big(a_{\nu-1}(x, y)\mathrm{d}y^{\nu-1}\mathrm{d}x + \cdots + a_0(x, y)\mathrm{d}x^\nu\Big) = 0\Big\} \qquad \text{and}$$

$$\mathrm{T}\mathcal{W}_{d-\nu-1}|_U = \left\{\prod_{l=1}^{d-\nu-1} (\mathrm{d}x + g_l(x, y)\mathrm{d}y) = 0\right\}.$$

Then, by passing to the ramified covering $\pi : (x, y) \mapsto (x, y^\nu)$, we obtain that $\pi^*\mathcal{F} = \mathcal{F}_0$, $\pi^*\mathcal{W}_\nu = \boxtimes_{k=1}^{\nu}\mathcal{F}_k$ and $\pi^*\mathcal{W}_{d-\nu-1} = \boxtimes_{l=1}^{d-\nu-1}\mathcal{F}_{\nu+l}$, where

$$\begin{aligned}
\mathcal{F}_0 &: \mathrm{d}y + \tfrac{1}{\nu} yh(x, y^\nu)\mathrm{d}x = 0,\\
\mathcal{F}_k &: \mathrm{d}x + y^{\nu-2} f(x, \zeta^k y)\zeta^{-k}\mathrm{d}y = 0,\\
\mathcal{F}_{\nu+l} &: \mathrm{d}x + \nu y^{\nu-1} g_l(x, y^\nu)\mathrm{d}y = 0,
\end{aligned}$$

with $\zeta = \exp(\frac{2\mathrm{i}\pi}{\nu})$. Therefore we have

$$K(\pi^*\mathcal{W}) = K\big(\pi^*(\mathcal{W}_\nu \boxtimes \mathcal{W}_{d-\nu-1})\big) + \sum_{1\leq i<j\leq d-1} K(\mathcal{F}_0 \boxtimes \mathcal{F}_i \boxtimes \mathcal{F}_j).$$

Now, on the one hand, [13, Proposition 2.6] ensures that $K(\mathcal{W}_\nu \boxtimes \mathcal{W}_{d-\nu-1})$ is holomorphic along $\{y = 0\}$; therefore so is $K\big(\pi^*(\mathcal{W}_\nu \boxtimes \mathcal{W}_{d-\nu-1})\big) = \pi^*\big(K(\mathcal{W}_\nu \boxtimes \mathcal{W}_{d-\nu-1})\big)$. On the other hand, since $\{y = 0\}$ is invariant by $\mathcal{F}_0$ and $\{y = 0\} \not\subset \mathrm{Tang}(\mathcal{F}_0, \mathcal{F}_i \boxtimes \mathcal{F}_j)$, then $K(\mathcal{F}_0 \boxtimes \mathcal{F}_i \boxtimes \mathcal{F}_j)$ is holomorphic on $\{y = 0\}$ by applying [13, Theorem 1], *see* also [2, Theorem 1.1] or [3, Corollary 1.30]. It follows that $\pi^*K(\mathcal{W}) = K(\pi^*\mathcal{W})$ is holomorphic on $\{y = 0\}$. As a consequence $K(\mathcal{W})$ is holomorphic along C. □

Remark 3.10 Similarly, we obtain an analogue of Proposition 3.9 by replacing the foliation $\mathcal{F}$ by a 2-web $\mathcal{W}_2 = \mathcal{F}_1 \boxtimes \mathcal{F}_2$ leaving the component $C \subset \Delta(\mathcal{W}_\nu)$ totally invariant.

Proof of Theorem 3.7 ***i.*** First assume that $\ell = L_\infty$. Then Theorem 3.1 ensures that $K(\mathrm{Leg}\mathscr{H}) = K(\mathrm{Leg}\mathcal{H})$. Now, we know from [4, Theorem 3.1] that the flatness of the web $\mathrm{Leg}\mathcal{H}$ is characterized by the holomorphy of its curvature $K(\mathrm{Leg}\mathcal{H})$ on $\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$. Therefore the same is true for the web $\mathrm{Leg}\mathscr{H}$, *i.e.* $\mathrm{Leg}\mathscr{H}$ is flat if and only if $K(\mathrm{Leg}\mathscr{H})$ is holomorphic along $\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$.

ii. Now assume that ℓ passes through the origin. Let us fix $s \in \Sigma^{\infty}_{\mathcal{H}}$ and describe the d-web $\mathrm{Leg}\mathscr{H}$ in a neighborhood of a generic point m of the line $\check{s}$ dual to s. Denote by $\nu - 1 \geq 0$ the radiality order of s; by [13, Proposition 3.3], in a neighborhood of m, we can decompose $\mathrm{Leg}\mathscr{H}$ as

$$\mathrm{Leg}\mathscr{H} = \mathrm{Leg}\ell \boxtimes \mathcal{W}_\nu \boxtimes \mathcal{W}_{d-\nu-1}, \tag{3.1}$$

where $\mathcal{W}_\nu$ is an irreducible ν-web leaving $\check{s}$ invariant and whose discriminant $\Delta(\mathcal{W}_\nu)$ has minimal multiplicity $\nu - 1$ along $\check{s}$, and where $\mathcal{W}_{d-\nu-1}$ is a $(d-\nu-1)$-web transverse to $\check{s}$. More explicitly, up to linear conjugation, we can write $\ell = (y = \alpha x)$, $s = [1 : \rho : 0]$, $\check{s} = \{p = \rho\}$, $m = (\rho, q)$ and $\underline{\mathcal{G}}^{-1}_{\mathcal{H}}(\rho) = \{\rho, r_1, \ldots, r_{d-\nu-1}\}$, so that (*see* [4, Section 2])

$$\mathrm{Leg}\ell : (p-\alpha)\mathrm{d}q - q\mathrm{d}p = 0, \qquad \mathcal{W}_\nu\Big|_{\check{s}} : \mathrm{d}p = 0, \qquad \mathcal{W}_{d-\nu-1}\Big|_{\check{s}} : \prod_{i=1}^{d-\nu-1}\Big((\rho - r_i)\mathrm{d}q - q\mathrm{d}p\Big) = 0.$$

We deduce, in particular, the two following properties:

(a) if $\check{s} \not\subset \mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$, the web $\mathcal{W}_{d-\nu-1}$ is regular in a neighborhood of m, because we then have $r_i \neq r_j$ if $i \neq j$;
(b) if $\check{s} \neq D_\ell = \{p = \underline{\mathcal{G}}_{\mathcal{H}}(\alpha)\}$, then $\mathrm{Leg}\ell$ is transverse to $\check{s}$ and $\check{s} \not\subset \mathrm{Tang}(\mathrm{Leg}\ell, \mathcal{W}_{d-\nu-1})$.

If $s \in \Sigma^{\mathrm{rad}}_{\mathcal{H}}$ is such that $\check{s} \not\subset \mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}) \cup D_\ell$, then properties (a) and (b) ensure that the $(d-\nu)$-web $\mathcal{W}_{d-\nu} := \mathrm{Leg}\ell \boxtimes \mathcal{W}_{d-\nu-1}$ is transverse to $\check{s}$ and is regular in a neighborhood of m. Therefore the curvature of $\mathrm{Leg}\mathscr{H} = \mathcal{W}_\nu \boxtimes \mathcal{W}_{d-\nu}$ is holomorphic in a neighborhood of m by applying [13, Proposition 2.6]. It follows that $K(\mathrm{Leg}\mathscr{H})$ is holomorphic on $\check{\Sigma}^{\mathrm{rad}}_{\mathcal{H}} \setminus (\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}) \cup D_\ell)$. Thus, according to the second assertion of Lemma 2.2 and Lemma 3.8, $\mathrm{Leg}\mathscr{H}$ is flat if and only if $K(\mathrm{Leg}\mathscr{H})$ is holomorphic along $\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}) \cup D_\ell$.

Let us show that in the particular case where ℓ is invariant by $\mathcal{H}$, the flatness of $\mathrm{Leg}\mathscr{H}$ is equivalent to the holomorphy of $K(\mathrm{Leg}\mathscr{H})$ on $\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$. From the above discussion, it suffices to prove that if D_ℓ is not contained in $\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$, then $K(\mathrm{Leg}\mathscr{H})$ is holomorphic on D_ℓ. The invariance of ℓ by $\mathcal{H}$ implies the existence of $s \in \Sigma^{\infty}_{\mathcal{H}}$ such that $\ell = (Os)$; then $D_\ell = \check{s}$ is invariant by the radial foliation $\mathrm{Leg}\ell$. Moreover, the condition $D_\ell \not\subset \mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$ implies that $\mathcal{W}_{d-\nu-1}$ is regular in a neighborhood of every generic point m of D_ℓ (property (a)). By applying Theorem 1 of [13] if $\nu = 1$ and Proposition 3.9 if $\nu \geq 2$, we deduce that $K(\mathrm{Leg}\mathscr{H})$ is holomorphic along D_ℓ. □

From Theorem 3.7 we deduce the two following corollaries.

Corollary 3.11 *Let $\mathscr{H}$ be a homogeneous convex pre-foliation of co-degree 1 and degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$. Then the d-web $\mathrm{Leg}\mathscr{H}$ is flat.*

Corollary 3.12 *Let $\mathscr{H} = \ell \boxtimes \mathcal{H}$ be a homogeneous pre-foliation of co-degree 1 and degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$. Assume that the homogeneous foliation $\mathcal{H}$ is convex and that the line ℓ is not invariant by $\mathcal{H}$. Then the d-web $\mathrm{Leg}\mathscr{H}$ is flat if and only if its curvature $K(\mathrm{Leg}\mathscr{H})$ is holomorphic on $D_\ell = \mathcal{G}_{\mathcal{H}}(\ell)$.*

The following theorem is an effective criterion for the holomorphy of the curvature of the web dual to a homogeneous pre-foliation $\mathscr{H} = \ell \boxtimes \mathcal{H}$ (with $O \in \ell$) along an irreducible component of $\Delta(\mathrm{Leg}\mathcal{H}) \setminus (D_\ell \cup \check{O})$.

Theorem 3.13 *Let $\mathscr{H} = \ell \boxtimes \mathcal{H}$ be a homogeneous pre-foliation of co-degree 1 and degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$, defined by the 1-form*

$$\omega = (\alpha\, x + \beta\, y)\,(A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y)\,, \quad A, B \in \mathbb{C}[x, y]_{d-1},\ \gcd(A, B) = 1.$$

Let (p, q) be the affine chart of $\check{\mathbb{P}}^2_{\mathbb{C}}$ associated to the line $\{y = px - q\} \subset \mathbb{P}^2_{\mathbb{C}}$ and let $D = \{p = p_0\}$ be an irreducible component of $\Delta(\mathrm{Leg}\mathcal{H}) \setminus (D_\ell \cup \check{O})$. Write $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0 : 1]) = \{[a_1 : b_1], \ldots, [a_n : b_n]\}$ and denote by ν_i the ramification index of $\underline{\mathcal{G}}_{\mathcal{H}}$ at the point $[a_i : b_i] \in \mathbb{P}^1_{\mathbb{C}}$. For $i \in \{1, \ldots, n\}$, define the polynomials $P_i \in \mathbb{C}[x, y]_{d-\nu_i-1}$ and $Q_i \in \mathbb{C}[x, y]_{2d-\nu_i-3}$ by

$$P_i(x, y; a_i, b_i) := \frac{\begin{vmatrix} A(x, y) & A(b_i, a_i) \\ B(x, y) & B(b_i, a_i) \end{vmatrix}}{(b_i\, y - a_i\, x)^{\nu_i}} \quad \text{and}$$

$$Q_i(x, y; a_i, b_i) := (\nu_i - 2)\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) P_i(x, y; a_i, b_i) + 2(\nu_i + 1)\begin{vmatrix} \frac{\partial P_i}{\partial x} & A(x, y) \\ \frac{\partial P_i}{\partial y} & B(x, y) \end{vmatrix}. \tag{3.2}$$

Then the curvature of $\mathrm{Leg}\mathscr{H}$ *is holomorphic on D if and only if*

$$\sum_{i=1}^{n}\left(1 - \frac{1}{\nu_i}\right)\left(p_0 b_i - a_i\right)\left(\frac{Q_i(b_i, a_i; a_i, b_i)}{B(b_i, a_i)P_i(b_i, a_i; a_i, b_i)} + \frac{3\nu_i(\alpha + p_0\,\beta)}{\alpha\, b_i + \beta\, a_i}\right) = 0.$$

The proof of this theorem is based on the criterion of [7, Theorem 2.1] for the holomorphy of the curvature of smooth webs. To do this, let us first recall the definition of the characteristic surface of a web and the definition of smooth web along an irreducible component of its discriminant. Let $\mathcal{W}$ be a holomorphic web on a complex surface M. Let $\widetilde{M} = \mathbb{P}\mathrm{T}^*M$ be the projectivization of the cotangent bundle of M; the *characteristic surface* of $\mathcal{W}$ is the surface $S_{\mathcal{W}} \subset \widetilde{M}$ defined by

$$S_{\mathcal{W}} := \overline{\left\{(m, [\eta]) \in \widetilde{M} \;\middle|\; m \in M \setminus \Delta(\mathcal{W}),\ \ker \eta \subset \mathrm{T}_m\mathcal{W}\right\}}$$

(*see* [7, §1.2] for a local expression of this surface). Denote by $\pi : \widetilde{M} \to M$ the natural projection and by $\pi_{\mathcal{W}} : S_{\mathcal{W}} \to M$ the restriction of π to $S_{\mathcal{W}}$. Let D be an irreducible component of the discriminant $\Delta(\mathcal{W})$. Following [7, Definition 1.1], the web $\mathcal{W}$ is said to be *smooth along D* if for every generic point m of D, the characteristic surface $S_{\mathcal{W}}$ of $\mathcal{W}$ is smooth at every point of the fiber $\pi_{\mathcal{W}}^{-1}(m)$.

Proof Let $\delta \in \mathbb{C}$ be such that $\beta + \alpha\delta \neq 0$ and $b_i - a_i\delta \neq 0$ for all $i = 1, \ldots, n$. Up to conjugating ω by the linear transformation $(x + \delta\, y, y)$, we can assume that none of the lines $\ell = (\alpha x + \beta y = 0)$ and $L_i = (b_i\, y - a_i\, x = 0)$ is vertical, *i.e.* that $\beta \neq 0$ and $b_i \neq 0$ for all $i = 1, \ldots, n$. Let us then put $\rho := -\frac{\alpha}{\beta}$ and $r_i := \frac{a_i}{b_i}$; we have $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}(p_0) = \{r_1, \ldots, r_n\}$ with $\underline{\mathcal{G}}_{\mathcal{H}}(z) = -\dfrac{A(1, z)}{B(1, z)}$. According to [7, Lemma 3.5], there therefore exists a constant $c \in \mathbb{C}^*$ such that

$$-A(1, z) = p_0 B(1, z) - c\prod_{i=1}^{n}(z - r_i)^{\nu_i}.$$

Moreover, the d-web $\mathrm{Leg}\mathscr{H}$ is given in the affine chart (p,q) by the differential equation

$$\Big((p-\rho)x-q\Big)\Big(A(x,px-q)+pB(x,px-q)\Big)=0, \qquad \text{with} \qquad x=\frac{\mathrm{d}q}{\mathrm{d}p}; \tag{3.3}$$

since $A, B \in \mathbb{C}[x,y]_{d-1}$, this equation can then be rewritten as

$$\begin{aligned}0&=x^{d-1}\Big((p-\rho)x-q\Big)\Big(A(1,p-\tfrac{q}{x})+pB(1,p-\tfrac{q}{x})\Big)\\&=x^{d}\Big(p-\frac{q}{x}-\rho\Big)\Big((p-p_0)B(1,p-\tfrac{q}{x})+c\prod_{i=1}^{n}(p-\tfrac{q}{x}-r_i)^{\nu_i}\Big), \qquad \text{with} \qquad x=\frac{\mathrm{d}q}{\mathrm{d}p}.\end{aligned}$$

Put $\check{x}:=q$, $\check{y}:=p-p_0$ and $\check{p}:=\dfrac{\mathrm{d}\check{y}}{\mathrm{d}\check{x}}=\dfrac{1}{x}$; in these new coordinates $D=\{\check{y}=0\}$ and $\mathrm{Leg}\mathscr{H}$ is described by the differential equation

$$F(\check{x},\check{y},\check{p}):=\Big(\check{y}+p_0-\check{p}\check{x}-\rho\Big)\Big(\check{y}B(1,\check{y}+p_0-\check{p}\check{x})+c\prod_{i=1}^{n}(\check{y}+p_0-\check{p}\check{x}-r_i)^{\nu_i}\Big)=0.$$

We have $F(\check{x},0,\check{p})=c(-\check{x})^d\big(\check{p}-\varphi_0(\check{x})\big)\prod_{i=1}^{n}\big(\check{p}-\varphi_i(\check{x})\big)^{\nu_i}$, where $\varphi_0(\check{x})=\dfrac{p_0-\rho}{\check{x}}$ and $\varphi_i(\check{x})=\dfrac{p_0-r_i}{\check{x}}$; the hypothesis that $D\neq D_\ell=\{p=\mathcal{G}_{\mathcal{H}}(\rho)\}$ translates into the fact that, for all $i\in\{1,\ldots,n\}$, $r_i\neq\rho$ and therefore $\varphi_i\not\equiv\varphi_0$. Note that if $\nu_i\geq 2$, then $\partial_{\check{y}}F\big(\check{x},0,\varphi_i(\check{x})\big)=(r_i-\rho)B(1,r_i)\neq 0$; since $\partial_{\check{p}}F\big(\check{x},0,\varphi_0(\check{x})\big)\not\equiv 0$ and $\partial_{\check{p}}F\big(\check{x},0,\varphi_i(\check{x})\big)\not\equiv 0$ if $\nu_i=1$, we deduce that the surface

$$S_{\mathrm{Leg}\mathscr{H}}=\Big\{(\check{x},\check{y},\check{p})\in\mathbb{P}\mathrm{T}^*\check{\mathbb{P}}^2_{\mathbb{C}}\mid F(\check{x},\check{y},\check{p})=0\Big\}$$

is smooth along $D=\{\check{y}=0\}$. Thus, according to [7, Theorem 2.1], the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic on $D=\{\check{y}=0\}$ if and only if $\sum_{i=1}^{n}(\nu_i-1)\varphi_i(\check{x})\psi_i(\check{x})\equiv 0$ and $\sum_{i=1}^{n}(\nu_i-1)\frac{\mathrm{d}}{\mathrm{d}\check{x}}\psi_i(\check{x})\equiv 0$, where, for all $i\in\{1,\ldots,n\}$ such that $\nu_i\geq 2$,

$$\begin{aligned}\psi_i(\check{x})=\frac{1}{\nu_i}\Bigg[&(\nu_i-2)\left(d-\varphi_i(\check{x})\frac{\partial_{\check{p}}\partial_{\check{y}}F\big(\check{x},0,\varphi_i(\check{x})\big)}{\partial_{\check{y}}F\big(\check{x},0,\varphi_i(\check{x})\big)}\right)\\&-2(\nu_i+1)\left(\frac{\varphi_0(\check{x})}{\varphi_i(\check{x})-\varphi_0(\check{x})}+\sum_{j=1,j\neq i}^{n}\frac{\nu_j\varphi_j(\check{x})}{\varphi_i(\check{x})-\varphi_j(\check{x})}\right)\Bigg].\end{aligned}$$

Now, if $\nu_i\geq 3$ then $\partial_{\check{p}}\partial_{\check{y}}F\big(\check{x},0,\varphi_i(\check{x})\big)=-\check{x}\Big(B(1,r_i)+(r_i-\rho)\partial_y B(1,r_i)\Big)$. It follows that

$$\begin{aligned}\psi_i(\check{x})=\psi_i:=\frac{1}{\nu_i}\Bigg[&(\nu_i-2)\left(d+\frac{\big(p_0-r_i\big)\Big(B(1,r_i)+(r_i-\rho)\partial_y B(1,r_i)\Big)}{(r_i-\rho)B(1,r_i)}\right)\\&+2(\nu_i+1)\left(\frac{p_0-\rho}{r_i-\rho}+\sum_{j=1,j\neq i}^{n}\frac{\nu_j(p_0-r_j)}{r_i-r_j}\right)\Bigg].\end{aligned}$$

Therefore $K(\mathrm{Leg}\mathscr{H})$ is holomorphic on $D=\{\check{y}=0\}$ if and only if $\sum_{i=1}^{n}(\nu_i-1)\varphi_i(\check{x})\psi_i\equiv 0$. On the other hand, arguing as in the proof of [7, Theorem 3.1], we obtain that

$$\sum_{j=1,j\neq i}^{n}\frac{\nu_j(p_0-r_j)}{r_i-r_j}=\frac{\begin{vmatrix}\partial_x P_i(1,r_i;r_i,1) & A(1,r_i)\\ \partial_y P_i(1,r_i;r_i,1) & B(1,r_i)\end{vmatrix}}{B(1,r_i)P_i(1,r_i;r_i,1)}$$

and that, for all $i\in\{1,\ldots,n\}$ such that $\nu_i\geq 2$,

$$(d-1)B(1,r_i)+(p_0-r_i)\partial_y B(1,r_i)=\partial_x B(1,r_i)-\partial_y A(1,r_i),$$

so that

$$\begin{aligned}\psi_i&=\frac{1}{\nu_i}\left[(\nu_i-2)\left(\frac{p_0-\rho}{r_i-\rho}+\frac{\partial_x B(1,r_i)-\partial_y A(1,r_i)}{B(1,r_i)}\right)\right.\\&\qquad\left.+2(\nu_i+1)\left(\frac{p_0-\rho}{r_i-\rho}+\frac{\begin{vmatrix}\partial_x P_i(1,r_i;r_i,1) & A(1,r_i)\\ \partial_y P_i(1,r_i;r_i,1) & B(1,r_i)\end{vmatrix}}{B(1,r_i)P_i(1,r_i;r_i,1)}\right)\right]\\&=\frac{Q_i(1,r_i;r_i,1)}{\nu_i B(1,r_i)P_i(1,r_i;r_i,1)}+\frac{3(p_0-\rho)}{r_i-\rho}.\end{aligned}$$

As a result, $K(\mathrm{Leg}\mathscr{H})$ is holomorphic along $D=\{\check{y}=0\}$ if and only if

$$\frac{1}{\check{x}}\sum_{i=1}^{n}\left(1-\frac{1}{\nu_i}\right)\left(p_0-r_i\right)\left(\frac{Q_i(1,r_i;r_i,1)}{B(1,r_i)P_i(1,r_i;r_i,1)}+\frac{3\nu_i(p_0-\rho)}{r_i-\rho}\right)=0,$$

hence the theorem follows. □

Remark 3.14 (i) We recover the fact (*cf.* step ***ii.*** of the proof of Theorem 3.7) that the curvature of $\mathrm{Leg}\mathscr{H}$ is always holomorphic along $\check{\Sigma}_{\mathcal{H}}^{\mathrm{rad}}\setminus(\mathcal{G}_{\mathcal{H}}(\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}})\cup D_\ell)$. Indeed, if D is contained in $\check{\Sigma}_{\mathcal{H}}^{\mathrm{rad}}\setminus(\mathcal{G}_{\mathcal{H}}(\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}})\cup D_\ell)$, then the fiber $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0:1])$ does not contain any non-fixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$, so that we have $p_0b_i-a_i=0$ if $\nu_i\geq 2$, which implies (Theorem 3.13) that $K(\mathrm{Leg}\mathscr{H})$ is holomorphic on D.

(ii) We know from [7, Theorem 3.1] that the curvature of $\mathrm{Leg}\mathcal{H}$ is holomorphic on D if and only if

$$\sum_{i=1}^{n}\left(1-\frac{1}{\nu_i}\right)\frac{(p_0b_i-a_i)Q_i(b_i,a_i;a_i,b_i)}{B(b_i,a_i)P_i(b_i,a_i;a_i,b_i)}=0.$$

From this result and Theorem 3.13, we deduce the following properties:

- If the curvature of $\mathrm{Leg}\mathcal{H}$ is holomorphic on D, then the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic on D if and only if

$$(\alpha+p_0\beta)\sum_{i=1}^{n}\frac{(\nu_i-1)(p_0b_i-a_i)}{\alpha b_i+\beta a_i}=0.$$

- In particular, when $d=3$ the fiber $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0:1])$ is reduced to a single point, say $[a:b]$, and the holomorphy of the curvature of $\mathrm{Leg}\mathscr{H}$ on D is equivalent to $(\alpha+p_0\beta)(p_0b-a)=0$, *i.e.* to $\alpha+p_0\beta=0$ or $[a:b]=[p_0:1]$, and therefore to $(1,p_0)\in\ell$ or $[p_0:1]$ is fixed by $\underline{\mathcal{G}}_{\mathcal{H}}$.

– If $(1, p_0) \in \ell$ then we have equivalence between the holomorphy on D of $K(\mathrm{Leg}\mathscr{H})$ and that of $K(\mathrm{Leg}\mathcal{H})$.

(iii) Assume that $\nu_i = \nu \geq 2$ for all $i \in \{1, \ldots, n\}$. Then the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic on D if and only if

$$\sum_{i=1}^{n} (p_0 b_i - a_i) \left(\frac{(\nu - 2)\big(\partial_x B(b_i, a_i) - \partial_y A(b_i, a_i)\big)}{B(b_i, a_i)} + \frac{3\nu(\alpha + p_0\,\beta)}{\alpha\, b_i + \beta\, a_i} \right) = 0.$$

Indeed, in the above proof, put $\delta_{i,j} = \frac{(p_0 - r_i)(p_0 - r_j)}{(r_i - r_j)}$ and note that

$$\sum_{i=1}^{n} \left((\nu - 1)\varphi_i(\check{x}) \sum_{j=1, j\neq i}^{n} \frac{\nu(p_0 - r_j)}{r_i - r_j} \right) = \frac{\nu(\nu-1)}{\check{x}} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \delta_{i,j} = \frac{\nu(\nu-1)}{\check{x}} \sum_{1\leq i<j\leq n} (\delta_{i,j} + \delta_{j,i}) \equiv 0.$$

In particular, if the fiber $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0 : 1])$ contains a single non-fixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$, say $[a : b]$, then

– either $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0 : 1]) = \{[a : b]\}$, in which case $\nu = d - 1$;
– or $\#\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([p_0 : 1]) = 2$, in which case d is necessarily odd, $d = 2k + 1$, and $\nu = k$.

In both cases, the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic on D if and only if

$$(\nu - 2)(\alpha\, b + \beta\, a)\Big(\partial_x B(b, a) - \partial_y A(b, a)\Big) + 3\nu(\alpha + p_0\,\beta)B(b, a) = 0.$$

Example 3.15 Let us consider the homogeneous pre-foliation $\mathscr{H} = \ell \boxtimes \mathcal{H}$ of co-degree 1 and odd degree $2k + 1 \geq 5$ on $\mathbb{P}^2_{\mathbb{C}}$ defined by the 1-form

$$\omega = (x - \tau\, y)\left(y^k (y - x)^k \mathrm{d}x + (y - \lambda\, x)^k (y - \mu\, x)^k \mathrm{d}y\right),$$
$$\text{where } \lambda, \mu \in \mathbb{C} \setminus \{0, 1\} \text{ and } \tau \in \mathbb{C} \setminus \{1\}.$$

We know from [7, Example 3.4] that $D := \{p = 0\} \subset \Delta(\mathrm{Leg}\mathcal{H})$ and that the fiber $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}([0 : 1])$ consists of the two points $[0 : 1]$ and $[1 : 1]$: the point $[0 : 1]$ (resp. $[1 : 1]$) is critical and fixed (resp. non-fixed) for $\underline{\mathcal{G}}_{\mathcal{H}}$ with multiplicity $k - 1$. Moreover, since $\tau \neq 1$, we have $[1 : \tau] \notin \underline{\mathcal{G}}_{\mathcal{H}}^{-1}([0 : 1])$, so that $D \neq D_\ell = \big\{[p : 1] = \underline{\mathcal{G}}_{\mathcal{H}}([1 : \tau])\big\}$. From Remark 3.14 (iii), we deduce that the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic along D if and only if

$$0 = (k-2)(1-\tau)\Big(\partial_x B(1,1) - \partial_y A(1,1)\Big) + 3kB(1,1) = k(1-\lambda)^k(1-\mu)^k \times \left(\frac{(k-2)(\tau-1)(\lambda+\mu-2\lambda\mu)}{(\lambda-1)(\mu-1)} + 3\right),$$

i.e. if and only if the quadruple (k, λ, μ, τ) satisfies the equation $(k-2)(\tau-1)(\lambda+\mu-2\lambda\mu) + 3(\lambda-1)(\mu-1) = 0.$ Note that, according to [7, Example 3.4], the holomorphy of the curvature of $\mathrm{Leg}\mathcal{H}$ along D is characterized by the equation $(k-2)(\lambda+\mu-2\lambda\mu) = 0$. It follows, in particular, that if the curvature of $\mathrm{Leg}\mathcal{H}$ is holomorphic on D, then the curvature of $\mathrm{Leg}\mathscr{H}$ is not holomorphic on D.

Corollary 3.16 *Let $\mathscr{H} = \ell \boxtimes \mathcal{H}$ be a homogeneous pre-foliation of co-degree* 1 *and degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$, defined by the* 1*-form*

$$\omega = (\alpha\, x + \beta\, y)\,(A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y)\,, \quad A, B \in \mathbb{C}[x, y]_{d-1}, \quad \gcd(A, B) = 1.$$

Assume that the foliation $\mathcal{H}$ *has a transverse inflection line* $T = (ax + by = 0)$ *of order* $\nu - 1$. *Assume moreover that* $[-a : b] \in \mathbb{P}^1_{\mathbb{C}}$ *is the only non-fixed critical point of* $\underline{\mathcal{G}}_{\mathcal{H}}$ *in its fiber* $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}(\underline{\mathcal{G}}_{\mathcal{H}}([-a : b]))$ *and that* $[-\alpha : \beta] \notin \underline{\mathcal{G}}_{\mathcal{H}}^{-1}(\underline{\mathcal{G}}_{\mathcal{H}}([-a : b]))$. *Then the curvature of* $\mathrm{Leg}\mathscr{H}$ *is holomorphic on* $T' = \mathcal{G}_{\mathcal{H}}(T)$ *if and only if*

$$(\alpha b - \beta a) Q(b, -a; a, b) + 3\nu\Big(\alpha B(b, -a) - \beta A(b, -a)\Big) P(b, -a; a, b) = 0,$$

where

$$Q(x, y; a, b) := (\nu - 2)\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) P(x, y; a, b) + 2(\nu + 1)\begin{vmatrix} \frac{\partial P}{\partial x} & A(x, y) \\ \frac{\partial P}{\partial y} & B(x, y) \end{vmatrix}$$

and

$$P(x, y; a, b) := \frac{\begin{vmatrix} A(x, y) & A(b, -a) \\ B(x, y) & B(b, -a) \end{vmatrix}}{(ax + by)^{\nu}}.$$

Proof Up to linear conjugation, we can assume that $T' \neq L_\infty$; then T' has the equation $p = p_0$, where $p_0 = -\frac{A(b,-a)}{B(b,-a)}$. According to Theorem 3.13, the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic on T' if and only if

$$\left(1 - \tfrac{1}{\nu}\right)(p_0 b + a)\left(\frac{Q(b, -a; a, b)}{B(b, -a) P(b, -a; a, b)} + \frac{3\nu(\alpha + p_0 \beta)}{\alpha b - \beta a}\right) = 0.$$

Now, the hypothesis that the point $[-a : b]$ is not fixed by $\underline{\mathcal{G}}_{\mathcal{H}}$ translates into $p_0 b + a \neq 0$. It follows that $K(\mathrm{Leg}\mathscr{H})$ is holomorphic on T' if and only if

$$\frac{Q(b, -a; a, b)}{P(b, -a; a, b)} + \frac{3\nu\big(\alpha B(b, -a) - \beta A(b, -a)\big)}{\alpha b - \beta a} = 0,$$

hence the corollary holds. □

In particular, we have:

Corollary 3.17 *Let* $\mathscr{H} = \ell \boxtimes \mathcal{H}$ *be a homogeneous pre-foliation of co-degree* 1 *and degree* $d \geq 3$ *on* $\mathbb{P}^2_{\mathbb{C}}$, *defined by the* 1*-form*

$$\omega = (\alpha x + \beta y)\,(A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y)\,, \quad A, B \in \mathbb{C}[x, y]_{d-1}, \ \gcd(A, B) = 1.$$

Assume that $\mathcal{H}$ *admits a transverse inflection line* $T = (ax + by = 0)$ *of maximal order* $d - 2$ *and that* $T \neq \ell$. *Then the curvature of* $\mathrm{Leg}\mathscr{H}$ *is holomorphic along* $T' = \mathcal{G}_{\mathcal{H}}(T)$ *if and only if*

$$(d - 3)(\alpha b - \beta a)\Big(\partial_x B(b, -a) - \partial_y A(b, -a)\Big) + 3(d - 1)\Big(\alpha B(b, -a) - \beta A(b, -a)\Big) = 0.$$

The following theorem is an effective criterion for the holomorphy of the curvature of the web dual to a homogeneous pre-foliation $\mathscr{H} = \ell \boxtimes \mathcal{H}$ (with $O \in \ell$) along the component $D_\ell \subset \Delta(\mathrm{Leg}\mathscr{H})$.

Theorem 3.18 *Let* $\mathscr{H} = \ell \boxtimes \mathcal{H}$ *be a homogeneous pre-foliation of co-degree* 1 *and degree* $d \geq 3$ *on* $\mathbb{P}^2_{\mathbb{C}}$, *defined by the* 1*-form*

$$\omega = (\alpha x + \beta y)\,(A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y)\,, \quad A, B \in \mathbb{C}[x, y]_{d-1}, \ \gcd(A, B) = 1.$$

Write $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}(\underline{\mathcal{G}}_{\mathcal{H}}([-\alpha:\beta]))=\{[-\alpha:\beta],[a_1:b_1],\ldots,[a_n:b_n]\}$ and denote by ν_i (resp. ν_0) the ramification index of $\underline{\mathcal{G}}_{\mathcal{H}}$ at the point $[a_i:b_i]$ (resp. $[-\alpha:\beta]$). Define the polynomials $P_0\in\mathbb{C}[x,y]_{d-\nu_0-1}$ and $Q_0\in\mathbb{C}[x,y]_{2d-\nu_0-3}$ by

$$P_0(x,y;\alpha,\beta):=\frac{\begin{vmatrix}A(x,y) & A(\beta,-\alpha)\\ B(x,y) & B(\beta,-\alpha)\end{vmatrix}}{(\alpha x+\beta y)^{\nu_0}}\quad \text{and}$$

$$Q_0(x,y;\alpha,\beta):=(\nu_0-1)\left(\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}\right)P_0(x,y;\alpha,\beta)+(2\nu_0+1)\begin{vmatrix}\frac{\partial P_0}{\partial x} & A(x,y)\\ \frac{\partial P_0}{\partial y} & B(x,y)\end{vmatrix}.$$

Assume that $\underline{\mathcal{G}}_{\mathcal{H}}([-\alpha:\beta])\neq\infty$ and let $p_0\in\mathbb{C}$ be such that $[p_0:1]=\underline{\mathcal{G}}_{\mathcal{H}}([-\alpha:\beta])$. Then the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic on D_ℓ if and only if

$$\left(1+\frac{1}{\nu_0}\right)\frac{(\alpha+p_0\beta)Q_0(\beta,-\alpha;\alpha,\beta)}{B(\beta,-\alpha)P_0(\beta,-\alpha;\alpha,\beta)}+\sum_{i=1}^{n}\left(1-\frac{1}{\nu_i}\right)\left(p_0b_i-a_i\right)$$
$$\left(\frac{Q_i(b_i,a_i;a_i,b_i)}{B(b_i,a_i)P_i(b_i,a_i;a_i,b_i)}+\frac{3\nu_i(\alpha+p_0\beta)}{\alpha b_i+\beta a_i}\right)=0,$$

where the P_i's and the Q_i's ($i=1,\ldots,n$) are the polynomials given by (3.2).

Note that the d-web $\mathrm{Leg}\mathscr{H}=\mathrm{Leg}\ell\boxtimes\mathrm{Leg}\mathcal{H}$ is not smooth along the component $D_\ell\subset\mathrm{Tang}(\mathrm{Leg}\ell,\mathrm{Leg}\mathcal{H})$ and therefore we cannot apply Theorem 2.1 of [7] to $\mathrm{Leg}\mathscr{H}$ as we did in the proof of Theorem 3.13. To prove Theorem 3.18, we will first establish, for a foliation $\mathcal{F}$ and a web $\mathcal{W}$ smooth along an irreducible component D of $\mathrm{Tang}(\mathcal{F},\mathcal{W})$, an effective criterion for the holomorphy of the curvature of $\mathcal{F}\boxtimes\mathcal{W}$ along D.

Theorem 3.19 *Let $\mathcal{W}$ be a holomorphic $(d-1)$-web on a complex surface M. Let $\mathcal{F}$ be a holomorphic foliation on M. Assume that $\mathcal{W}$ is smooth along an irreducible component D of $\mathrm{Tang}(\mathcal{F},\mathcal{W})$. Then the fundamental form $\eta(\mathcal{F}\boxtimes\mathcal{W})$ has simple poles along D. More precisely, choose a local coordinate system (x,y) on M such that $D=\{y=0\}$ and let $F(x,y,p)=0$, $p=\frac{\mathrm{d}y}{\mathrm{d}x}$, be an implicit differential equation defining $\mathcal{W}$. Write $F(x,0,p)=a_0(x)\prod_{\alpha=1}^{n}(p-\varphi_\alpha(x))^{\nu_\alpha}$, with $\varphi_\alpha\not\equiv\varphi_\beta$ if $\alpha\neq\beta$, and assume that $\mathcal{F}$ is given by a 1-form ω of type $\omega=\mathrm{d}y-(\varphi_1(x)+yf(x,y))\,\mathrm{d}x$. Define $Q(x,p)$ by $F(x,0,p)=(p-\varphi_1(x))^{\nu_1}Q(x,p)$ and put*

$$h(x)=\frac{1}{\nu_1}\Bigg[(\nu_1-1)\left(d-1-\varphi_1(x)\frac{\partial_p\partial_yF(x,0,\varphi_1(x))+2\delta_{\nu_1,2}f(x,0)Q(x,\varphi_1(x))}{\partial_yF(x,0,\varphi_1(x))}\right)$$
$$-(2\nu_1+1)\sum_{\alpha=2}^{n}\frac{\nu_\alpha\varphi_\alpha(x)}{\varphi_1(x)-\varphi_\alpha(x)}\Bigg]$$

(where $\delta_{\nu_1,2}=1$ if $\nu_1=2$ and 0 otherwise). Let ψ_α be a function of the coordinate x defined, for all $\alpha\in\{1,\ldots,n\}$ such that $\nu_\alpha\geq 2$, by

$$\psi_\alpha(x)=\frac{1}{\nu_\alpha}\Bigg[(\nu_\alpha-2)\left(d-1-\varphi_\alpha(x)\frac{\partial_p\partial_yF\big(x,0,\varphi_\alpha(x)\big)}{\partial_yF\big(x,0,\varphi_\alpha(x)\big)}\right)$$
$$-2(\nu_\alpha+1)\sum_{\beta=1,\beta\neq\alpha}^{n}\frac{\nu_\beta\varphi_\beta(x)}{\varphi_\alpha(x)-\varphi_\beta(x)}\Bigg].$$

Then the 1*-form* $\eta(\mathcal{F}\boxtimes\mathcal{W})-\frac{\theta}{6y}$ *is holomorphic along* $D=\{y=0\}$, *where*

$$\theta=(\nu_1+1)\Big[h(x)\big(\mathrm{d}y-\varphi_1(x)\mathrm{d}x\big)+(\nu_1-1)\mathrm{d}y\Big]\\+\sum_{\alpha=2}^{n}(\nu_\alpha-1)\left[\left(\psi_\alpha(x)+\frac{3\varphi_1(x)}{\varphi_1(x)-\varphi_\alpha(x)}\right)\big(\mathrm{d}y-\varphi_\alpha(x)\mathrm{d}x\big)+(\nu_\alpha-2)\mathrm{d}y\right].$$

In particular, the curvature $K(\mathcal{F}\boxtimes\mathcal{W})$ *is holomorphic along* D *if and only if*

$$(\nu_1+1)\varphi_1(x)h(x)+\sum_{\alpha=2}^{n}(\nu_\alpha-1)\varphi_\alpha(x)\left(\psi_\alpha(x)+\frac{3\varphi_1(x)}{\varphi_1(x)-\varphi_\alpha(x)}\right)\equiv 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((\nu_1+1)h(x)+\sum_{\alpha=2}^{n}(\nu_\alpha-1)\left(\psi_\alpha(x)+\frac{3\varphi_1(x)}{\varphi_1(x)-\varphi_\alpha(x)}\right)\right)\equiv 0.$$

Proof In a neighborhood of a generic point m of D, the web $\mathcal{W}$ decomposes as $\mathcal{W}=\boxtimes_{\alpha=1}^{n}\mathcal{W}_\alpha$, where $\mathcal{W}_\alpha=\boxtimes_{i=1}^{\nu_\alpha}\mathcal{F}_i^\alpha$ and $\mathcal{F}_i^\alpha|_{y=0}:\mathrm{d}y-\varphi_\alpha(x)\mathrm{d}x=0$. Then $\eta(\mathcal{F}\boxtimes\mathcal{W})=\eta(\mathcal{W})+\eta_1+\eta_2+\eta_3+\eta_4$, where

$$\eta_1=\sum_{1\leq i<j\leq\nu_1}\eta(\mathcal{F}\boxtimes\mathcal{F}_i^1\boxtimes\mathcal{F}_j^1),\ \eta_2=\sum_{\alpha=2}^{n}\sum_{\substack{1\leq i\leq\nu_1\\1\leq j\leq\nu_\alpha}}\eta(\mathcal{F}\boxtimes\mathcal{F}_i^1\boxtimes\mathcal{F}_j^\alpha),$$

$$\eta_3=\sum_{\alpha=2}^{n}\sum_{1\leq i<j\leq\nu_\alpha}\eta(\mathcal{F}\boxtimes\mathcal{F}_i^\alpha\boxtimes\mathcal{F}_j^\alpha),\ \eta_4=\sum_{2\leq\alpha<\beta\leq n}\sum_{\substack{1\leq i\leq\nu_\alpha\\1\leq j\leq\nu_\beta}}\eta(\mathcal{F}\boxtimes\mathcal{F}_i^\alpha\boxtimes\mathcal{F}_j^\beta).$$

According to [7, Theorem 2.1], the principal part of the Laurent series of $\eta(\mathcal{W})$ at $y=0$ is given by $\frac{\theta_0}{y}$, where

$$\theta_0=\frac{1}{6}\sum_{\alpha=1}^{n}\big(\nu_\alpha-1\big)\Big[\psi_\alpha(x)\big(\mathrm{d}y-\varphi_\alpha(x)\mathrm{d}x\big)+\big(\nu_\alpha-2\big)\mathrm{d}y\Big].$$

As for the 1-forms $\eta_1,\ldots,\eta_4$, first note that, as in the proof of [7, Theorem 2.1], the slope p_j $(j=1,\ldots,\nu_\alpha)$ of $\mathrm{T}_{(x,y)}\mathcal{F}_j^\alpha$ can be written as

$$p_j=\lambda_{\alpha,j}(x,y):=\varphi_\alpha(x)+\sum_{k\geq1}f_{\alpha,k}(x)\zeta_\alpha^{jk}y^{\frac{k}{\nu_\alpha}},\quad\text{where } f_{\alpha,k}\in\mathbb{C}\{x\},$$

with $f_{\alpha,1}\not\equiv 0$ and $\zeta_\alpha=\exp(\frac{2i\pi}{\nu_\alpha})$. Moreover, for $\alpha=1$, if $\nu_1\geq 2$, then

$$(f_{1,1}(x))^{\nu_1}=-\frac{\partial_y F\big(x,0,\varphi_1(x)\big)}{Q(x,\varphi_1(x))}\tag{3.4}$$

and, for all $\alpha\in\{1,\ldots,n\}$ such that $\nu_\alpha\geq 2$, we have

$$\frac{f_{\alpha,2}(x)}{(f_{\alpha,1}(x))^2}=\frac{1}{\nu_\alpha}\left[\frac{\partial_p\partial_y F\big(x,0,\varphi_\alpha(x)\big)}{\partial_y F\big(x,0,\varphi_\alpha(x)\big)}-\sum_{\beta=1,\beta\neq\alpha}^{n}\frac{\nu_\beta}{\varphi_\alpha(x)-\varphi_\beta(x)}\right].\tag{3.5}$$

Put $\lambda_0(x, y) = \varphi_1(x) + yf(x, y)$; according to [7, Lemma 2.8], we have $\eta(\mathcal{F} \boxtimes \mathcal{F}_i^1 \boxtimes \mathcal{F}_j^1) = a_{i,j}(x, y)\mathrm{d}x + b_{i,j}(x, y)\mathrm{d}y$, where

$$a_{i,j} = -\frac{\left(\partial_y(\lambda_{1,i}\lambda_{1,j}) - \partial_x\lambda_0\right)\lambda_0}{(\lambda_{1,i} - \lambda_0)(\lambda_{1,j} - \lambda_0)} - \frac{\left(\partial_y(\lambda_{1,i}\lambda_0) - \partial_x\lambda_{1,j}\right)\lambda_{1,j}}{(\lambda_{1,i} - \lambda_{1,j})(\lambda_0 - \lambda_{1,j})} - \frac{\left(\partial_y(\lambda_{1,j}\lambda_0) - \partial_x\lambda_{1,i}\right)\lambda_{1,i}}{(\lambda_{1,j} - \lambda_{1,i})(\lambda_0 - \lambda_{1,i})}$$

and

$$b_{i,j} = \frac{\partial_y(\lambda_{1,i}\lambda_{1,j}) - \partial_x\lambda_0}{(\lambda_{1,i} - \lambda_0)(\lambda_{1,j} - \lambda_0)} + \frac{\partial_y(\lambda_{1,i}\lambda_0) - \partial_x\lambda_{1,j}}{(\lambda_{1,i} - \lambda_{1,j})(\lambda_0 - \lambda_{1,j})} + \frac{\partial_y(\lambda_{1,j}\lambda_0) - \partial_x\lambda_{1,i}}{(\lambda_{1,j} - \lambda_{1,i})(\lambda_0 - \lambda_{1,i})}.$$

Writing $f(x, y) = \sum_{k\geq 0} f_{0,k}(x)y^k$ and putting $w_1 = y^{\frac{1}{\nu_1}}$, a straightforward computation leads to the following equalities:

$$\partial_y(\lambda_{1,i}\lambda_{1,j}) - \partial_x\lambda_0 = \frac{1}{\nu_1 y}\Big[(\zeta_1^i + \zeta_1^j)\varphi_1 f_{1,1}w_1 + 2\left(\zeta_1^{i+j}f_{1,1}^2 + (\zeta_1^{2i} + \zeta_1^{2j})\varphi_1 f_{1,2} - \delta_{\nu_1,2}\varphi_1'\right)w_1^2 + \cdots\Big],$$

$$\partial_y(\lambda_{1,i}\lambda_0) - \partial_x\lambda_{1,j} = \frac{1}{\nu_1 y}\left[\zeta_1^i\varphi_1 f_{1,1}w_1 + 2\left(\zeta_1^{2i}\varphi_1 f_{1,2} + (\varphi_1 f_{0,0} - \varphi_1')\delta_{\nu_1,2}\right)w_1^2 + \cdots\right],$$

$$(\lambda_{1,i} - \lambda_0)(\lambda_{1,j} - \lambda_0) = \zeta_1^{i+j}f_{1,1}^2 w_1^2 + (\zeta_1^{2i+j} + \zeta_1^{i+2j})f_{1,1}f_{1,2}w_1^3 + \cdots,$$

$$(\lambda_{1,i} - \lambda_{1,j})(\lambda_0 - \lambda_{1,j}) = (\zeta_1^{2j} - \zeta_1^{i+j})f_{1,1}^2 w_1^2 + f_{1,1}\left((2\zeta_1^{3j} - \zeta_1^{2i+j} - \zeta_1^{i+2j})f_{1,2} - 2\delta_{\nu_1,2}f_{0,0}\right)w_1^3 + \cdots.$$

These equalities allow us to check that $a_{i,j}$ and $b_{i,j}$ can be written as

$$a_{i,j} = \frac{\varphi_1\left(\varphi_1 f_{1,2} - f_{1,1}^2 - \delta_{\nu_1,2}\varphi_1 f_{0,0}\right) + w_1 A_{i,j}}{\nu_1 y f_{1,1}^2},$$

$$b_{i,j} = \frac{2f_{1,1}^2 - \varphi_1 f_{1,2} + \delta_{\nu_1,2}\varphi_1 f_{0,0} + w_1 B_{i,j}}{\nu_1 y f_{1,1}^2},$$

where $A_{i,j}, B_{i,j} \in \mathbb{C}\{x, w_1\}$. Since η_1 is a uniform and meromorphic 1-form, we deduce that the principal part of the LAURENT series of η_1 at $y = 0$ is given by $\frac{\theta_1}{y}$, where

$$\theta_1 = \binom{\nu_1}{2}\left(\frac{\varphi_1(x)\Big(\varphi_1(x)f_{1,2}(x) - f_{1,1}(x)^2 - \delta_{\nu_1,2}\varphi_1(x)f_{0,0}(x)\Big)}{\nu_1 f_{1,1}(x)^2}\mathrm{d}x + \frac{2f_{1,1}(x)^2 - \varphi_1(x)f_{1,2}(x) + \delta_{\nu_1,2}\varphi_1(x)f_{0,0}(x)}{\nu_1 f_{1,1}(x)^2}\mathrm{d}y\right)$$
$$= \frac{1}{2}(\nu_1 - 1)\left[\varphi_1(x)\left(\frac{\delta_{\nu_1,2}f_{0,0}(x)}{f_{1,1}(x)^2} - \frac{f_{1,2}(x)}{f_{1,1}(x)^2}\right)\left(\mathrm{d}y - \varphi_1(x)\mathrm{d}x\right) + 2\mathrm{d}y - \varphi_1(x)\mathrm{d}x\right].$$

Thanks to (3.4), (3.5) and the equality $f_{0,0}(x) = f(x, 0)$, the 1-form θ_1 can be rewritten as

$$\theta_1 = \frac{1}{2}\left(1 - \frac{1}{\nu_1}\right)\left(d - 1 - \varphi_1(x)\frac{\partial_p\partial_y F(x, 0, \varphi_1(x)) + 2\delta_{\nu_1,2}f(x, 0)Q(x, \varphi_1(x))}{\partial_y F(x, 0, \varphi_1(x))} + \sum_{\alpha=2}^{n}\frac{\nu_\alpha\varphi_\alpha(x)}{\varphi_1(x) - \varphi_\alpha(x)}\right)\left(\mathrm{d}y - \varphi_1(x)\mathrm{d}x\right) + \frac{1}{2}(\nu_1 - 1)\mathrm{d}y.$$

Let us now pass to η_2. Put $w_{\alpha,1} = y^{\frac{1}{\nu_1\nu_\alpha}}$; again by [7, Lemma 2.8], we have $\eta(\mathcal{F} \boxtimes \mathcal{F}_i^1 \boxtimes \mathcal{F}_j^\alpha) = a_{i,j}^\alpha(x, y)\mathrm{d}x + b_{i,j}^\alpha(x, y)\mathrm{d}y$, where

$$a_{i,j}^\alpha = -\frac{\left(\partial_y(\lambda_{1,i}\lambda_{\alpha,j}) - \partial_x\lambda_0\right)\lambda_0}{(\lambda_{1,i} - \lambda_0)(\lambda_{\alpha,j} - \lambda_0)} - \frac{\left(\partial_y(\lambda_{1,i}\lambda_0) - \partial_x\lambda_{\alpha,j}\right)\lambda_{\alpha,j}}{(\lambda_{1,i} - \lambda_{\alpha,j})(\lambda_0 - \lambda_{\alpha,j})} - \frac{\left(\partial_y(\lambda_{\alpha,j}\lambda_0) - \partial_x\lambda_{1,i}\right)\lambda_{1,i}}{(\lambda_{\alpha,j} - \lambda_{1,i})(\lambda_0 - \lambda_{1,i})}$$

$$= \frac{1}{\nu_1 y}\left(\frac{\varphi_1\varphi_\alpha}{\varphi_1-\varphi_\alpha}+w_{\alpha,1}A^\alpha_{i,j}\right)$$

and

$$\begin{aligned} b^\alpha_{i,j} &= \frac{\partial_y(\lambda_{1,i}\lambda_{\alpha,j})-\partial_x\lambda_0}{(\lambda_{1,i}-\lambda_0)(\lambda_{\alpha,j}-\lambda_0)}+\frac{\partial_y(\lambda_{1,i}\lambda_0)-\partial_x\lambda_{\alpha,j}}{(\lambda_{1,i}-\lambda_{\alpha,j})(\lambda_0-\lambda_{\alpha,j})}+\frac{\partial_y(\lambda_{\alpha,j}\lambda_0)-\partial_x\lambda_{1,i}}{(\lambda_{\alpha,j}-\lambda_{1,i})(\lambda_0-\lambda_{1,i})}\\ &= -\frac{1}{\nu_1 y}\left(\frac{\varphi_\alpha}{\varphi_1-\varphi_\alpha}+w_{\alpha,1}B^\alpha_{i,j}\right), \end{aligned}$$

where $A^\alpha_{i,j}, B^\alpha_{i,j} \in \mathbb{C}\{x, w_{\alpha,1}\}$. The 1-form η_2 being uniform and meromorphic, it follows that the principal part of the LAURENT series of η_2 at $y = 0$ is given by $\frac{\theta_2}{y}$, where

$$\begin{aligned} \theta_2 &= \sum_{\alpha=2}^{n}\nu_1\nu_\alpha\left(\frac{\varphi_1(x)\varphi_\alpha(x)}{\nu_1\left(\varphi_1(x)-\varphi_\alpha(x)\right)}\mathrm{d}x-\frac{\varphi_\alpha(x)}{\nu_1\left(\varphi_1(x)-\varphi_\alpha(x)\right)}\mathrm{d}y\right)\\ &= -(\mathrm{d}y-\varphi_1(x)\mathrm{d}x)\sum_{\alpha=2}^{n}\frac{\nu_\alpha\varphi_\alpha(x)}{\varphi_1(x)-\varphi_\alpha(x)}. \end{aligned}$$

Similarly, putting $w_\alpha = y^{\frac{1}{\nu_\alpha}}$ and using [7, Lemma 2.8], we obtain that

$$\begin{aligned} \eta(\mathcal{F}\boxtimes\mathcal{F}^\alpha_i\boxtimes\mathcal{F}^\alpha_j) = \frac{1}{\nu_\alpha y}\Bigg[&\left(-\frac{\varphi_1(x)\varphi_\alpha(x)}{\varphi_1(x)-\varphi_\alpha(x)}+w_\alpha\,\tilde{A}^\alpha_{i,j}(x,w_\alpha)\right)\mathrm{d}x\\ &+\left(\frac{\varphi_1(x)}{\varphi_1(x)-\varphi_\alpha(x)}+w_\alpha\,\tilde{B}^\alpha_{i,j}(x,w_\alpha)\right)\mathrm{d}y\Bigg], \end{aligned}$$

where $\tilde{A}^\alpha_{i,j}, \tilde{B}^\alpha_{i,j} \in \mathbb{C}\{x, w_\alpha\}$, so that the principal part of the LAURENT series of η_3 at $y = 0$ is given by $\frac{\theta_3}{y}$, where

$$\begin{aligned} \theta_3 &= \sum_{\alpha=2}^{n}\binom{\nu_\alpha}{2}\left(-\frac{\varphi_1(x)\varphi_\alpha(x)}{\nu_\alpha\left(\varphi_1(x)-\varphi_\alpha(x)\right)}\mathrm{d}x+\frac{\varphi_1(x)}{\nu_\alpha\left(\varphi_1(x)-\varphi_\alpha(x)\right)}\mathrm{d}y\right)\\ &= \frac{1}{2}\sum_{\alpha=2}^{n}\frac{(\nu_\alpha-1)\varphi_1(x)\,(\mathrm{d}y-\varphi_\alpha(x)\mathrm{d}x)}{\varphi_1(x)-\varphi_\alpha(x)}. \end{aligned}$$

Finally, since $(\varphi_1-\varphi_\alpha)(\varphi_\alpha-\varphi_\beta)(\varphi_\beta-\varphi_1) \not\equiv 0$ for all $\beta > \alpha \geq 2$, [7, Lemma 2.8] implies that the 1-form $\eta(\mathcal{F}\boxtimes\mathcal{F}^\alpha_i\boxtimes\mathcal{F}^\beta_j)$ has no poles along $y = 0$; therefore the same is true for the 1-form η_4.

As a result, the principal part of the LAURENT series of $\eta(\mathcal{F}\boxtimes\mathcal{W})$ at $y = 0$ is given by $\frac{\widehat{\theta}}{y}$, where

$$\begin{aligned} \widehat{\theta} =&\theta_0+\theta_1+\theta_2+\theta_3\\ =&\frac{1}{6}\Big((\nu_1+1)h(x)-(\nu_1-1)\psi_1(x)\Big)\Big(\mathrm{d}y-\varphi_1(x)\mathrm{d}x\Big)+\frac{1}{2}(\nu_1-1)\mathrm{d}y\\ &+\frac{1}{6}\sum_{\alpha=1}^{n}(\nu_\alpha-1)\Big[\psi_\alpha(x)\big(\mathrm{d}y-\varphi_\alpha(x)\mathrm{d}x\big)+\big(\nu_\alpha-2\big)\mathrm{d}y\Big]\\ &+\frac{1}{2}\sum_{\alpha=2}^{n}\frac{(\nu_\alpha-1)\varphi_1(x)\,(\mathrm{d}y-\varphi_\alpha(x)\mathrm{d}x)}{\varphi_1(x)-\varphi_\alpha(x)}\\ =&\frac{1}{6}(\nu_1+1)\Big[h(x)\Big(\mathrm{d}y-\varphi_1(x)\mathrm{d}x\Big)+(\nu_1-1)\mathrm{d}y\Big] \end{aligned}$$

$$+\frac{1}{6}\sum_{\alpha=2}^{n}(\nu_\alpha-1)\left[\left(\psi_\alpha(x)+\frac{3\varphi_1(x)}{\varphi_1(x)-\varphi_\alpha(x)}\right)\big(\mathrm{d}y-\varphi_\alpha(x)\mathrm{d}x\big)+(\nu_\alpha-2)\mathrm{d}y\right]$$
$$=\frac{\theta}{6},$$

hence the theorem follows. □

Proof of Theorem 3.18 As in the proof of Theorem 3.13, up to linear conjugation, we can assume that $\beta\neq 0$ and $b_i\neq 0$ for all $i\in\{1,\ldots,n\}$. Then, by putting $r_0:=-\frac{\alpha}{\beta}$ and $r_i:=\frac{a_i}{b_i}$ for $i\in\{1,\ldots,n\}$, [7, Lemma 3.5] implies the existence of a constant $c\in\mathbb{C}^*$ such that

$$-A(1,z)=p_0B(1,z)-c\prod_{i=0}^{n}(z-r_i)^{\nu_i}.$$

Since $A,B\in\mathbb{C}[x,y]_{d-1}$, the differential equation (3.3) describing $\mathrm{Leg}\mathscr{H}$ in the affine chart (p,q) then becomes

$$x^d\left(p-\tfrac{q}{x}-r_0\right)\left((p-p_0)B(1,p-\tfrac{q}{x})+c\prod_{i=0}^{n}(p-\tfrac{q}{x}-r_i)^{\nu_i}\right)=0,\qquad\text{with}\qquad x=\frac{\mathrm{d}q}{\mathrm{d}p}.$$

Put $\check{x}:=q$, $\check{y}:=p-p_0$ and $\check{p}:=\dfrac{\mathrm{d}\check{y}}{\mathrm{d}\check{x}}=\dfrac{1}{x}$; in this new coordinate system $D_\ell=\{\check{y}=0\}$ and $\mathrm{Leg}\mathscr{H}=\mathrm{Leg}\ell\boxtimes\mathrm{Leg}\mathcal{H}$ is given by the differential equation $(\check{y}+p_0-\check{p}\check{x}-r_0)F(\check{x},\check{y},\check{p})=0$, where

$$F(\check{x},\check{y},\check{p})=\check{y}B(1,\check{y}+p_0-\check{p}\check{x})+c\prod_{i=0}^{n}(\check{y}+p_0-\check{p}\check{x}-r_i)^{\nu_i}.$$

We have $F(\check{x},0,\check{p})=c(-\check{x})^{d-1}\prod_{i=0}^{n}\big(\check{p}-\varphi_i(\check{x})\big)^{\nu_i}$, where $\varphi_i(\check{x})=\frac{p_0-r_i}{\check{x}}$. Furthermore the radial foliation $\mathrm{Leg}\ell$ is described by $\check{\omega}_0=\mathrm{d}\check{y}-\big(\varphi_0(\check{x})+\frac{\check{y}}{\check{x}}\big)\mathrm{d}\check{x}$; in particular we have $D_\ell\subset\mathrm{Tang}(\mathrm{Leg}\ell,\mathrm{Leg}\mathcal{H})$. Note that if $\nu_i\geq 2$, then $\partial_{\check{y}}F\big(\check{x},0,\varphi_i(\check{x})\big)=B(1,r_i)\neq 0$; since $\partial_{\check{p}}F\big(\check{x},0,\varphi_i(\check{x})\big)\not\equiv 0$ if $\nu_i=1$, it follows that the surface

$$S_{\mathrm{Leg}\mathcal{H}}=\left\{(\check{x},\check{y},\check{p})\in\mathbb{P}\mathrm{T}^*\check{\mathbb{P}}^2_{\mathbb{C}}\mid F(\check{x},\check{y},\check{p})=0\right\}$$

is smooth along $D_\ell=\{\check{y}=0\}$. Therefore, according to Theorem 3.19, the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic on $D_\ell=\{\check{y}=0\}$ if and only if

$$(\nu_0+1)\varphi_0(\check{x})h(\check{x})+\sum_{i=1}^{n}(\nu_i-1)\varphi_i(\check{x})\left(\psi_i(\check{x})+\frac{3\varphi_0(\check{x})}{\varphi_0(\check{x})-\varphi_i(\check{x})}\right)\equiv 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\check{x}}\left((\nu_0+1)h(\check{x})+\sum_{i=1}^{n}(\nu_i-1)\left(\psi_i(\check{x})+\frac{3\varphi_0(\check{x})}{\varphi_0(\check{x})-\varphi_i(\check{x})}\right)\right)\equiv 0,$$

where

$$h(\check{x})=\frac{1}{\nu_0}\left[(\nu_0-1)\left(d-1-\varphi_0(\check{x})\frac{\partial_{\check{p}}\partial_{\check{y}}F\big(\check{x},0,\varphi_0(\check{x})\big)-2c\delta_{\nu_0,2}(-\check{x})^{d-2}\prod_{j=1}^{n}\big(\varphi_0(\check{x})-\varphi_j(\check{x})\big)^{\nu_j}}{\partial_{\check{y}}F\big(\check{x},0,\varphi_0(\check{x})\big)}\right)\right.$$
$$\left.-(2\nu_0+1)\sum_{j=1}^{n}\frac{\nu_j\varphi_j(\check{x})}{\varphi_0(\check{x})-\varphi_j(\check{x})}\right]$$

and, for all $i \in \{1, \ldots, n\}$ such that $\nu_i \geq 2$,

$$\psi_i(\check{x}) = \frac{1}{\nu_i}\left[(\nu_i - 2)\left(d - 1 - \varphi_i(\check{x})\frac{\partial_{\check{p}}\partial_{\check{y}}F\big(\check{x}, 0, \varphi_i(\check{x})\big)}{\partial_{\check{y}}F\big(\check{x}, 0, \varphi_i(\check{x})\big)}\right) - 2(\nu_i + 1)\sum_{j=0, j\neq i}^{n}\frac{\nu_j\varphi_j(\check{x})}{\varphi_i(\check{x}) - \varphi_j(\check{x})}\right].$$

Now, if $\nu_i \geq 2$ then $\partial_{\check{p}}\partial_{\check{y}}F\big(\check{x}, 0, \varphi_i(\check{x})\big) = -\check{x}\Big(\partial_y B(1, r_i) + 2c\delta_{\nu_i,2}\prod_{j=0, j\neq i}^{n}\big(r_i - r_j\big)^{\nu_j}\Big)$. From this we deduce that

$$h(\check{x}) = h_0 := \frac{1}{\nu_0}\left[(\nu_0 - 1)\left(d - 1 + \frac{(p_0 - r_0)\partial_y B(1, r_0)}{B(1, r_0)}\right) + (2\nu_0 + 1)\sum_{j=1}^{n}\frac{\nu_j(p_0 - r_j)}{r_0 - r_j}\right]$$

and

$$\psi_i(\check{x}) = \psi_i := \frac{1}{\nu_i}\left[(\nu_i - 2)\left(d - 1 + \frac{(p_0 - r_i)\partial_y B(1, r_i)}{B(1, r_i)}\right) + 2(\nu_i + 1)\sum_{j=0, j\neq i}^{n}\frac{\nu_j(p_0 - r_j)}{r_i - r_j}\right].$$

Thus $K(\mathrm{Leg}\mathscr{H})$ is holomorphic along $D_\ell = \{\check{y} = 0\}$ if and only if

$$(\nu_0 + 1)(p_0 - r_0)h_0 + \sum_{i=1}^{n}(\nu_i - 1)(p_0 - r_i)\left(\psi_i + \tfrac{3(p_0 - r_0)}{r_i - r_0}\right) = 0.$$

Moreover, we have (*cf.* proof of [7, Theorem 3.1])

$$\sum_{j=1}^{n}\frac{\nu_j(p_0 - r_j)}{r_0 - r_j} = \frac{\begin{vmatrix}\partial_x P_0(1, r_0; -r_0, 1) & A(1, r_0)\\ \partial_y P_0(1, r_0; -r_0, 1) & B(1, r_0)\end{vmatrix}}{B(1, r_0)P_0(1, r_0; -r_0, 1)},$$

$$\sum_{j=0, j\neq i}^{n}\frac{\nu_j(p_0 - r_j)}{r_i - r_j} = \frac{\begin{vmatrix}\partial_x P_i(1, r_i; r_i, 1) & A(1, r_i)\\ \partial_y P_i(1, r_i; r_i, 1) & B(1, r_i)\end{vmatrix}}{B(1, r_i)P_i(1, r_i; r_i, 1)} \quad (\text{for } i = 1, \ldots, n)$$

and, for all $i \in \{0, \ldots, n\}$ such that $\nu_i \geq 2$,

$$(d - 1)B(1, r_i) + (p_0 - r_i)\partial_y B(1, r_i) = \partial_x B(1, r_i) - \partial_y A(1, r_i).$$

By the definition of the polynomials Q_i's, it follows that

$$h_0 = \frac{Q_0(1, r_0; -r_0, 1)}{\nu_0 B(1, r_0)P_0(1, r_0; -r_0, 1)} \qquad \text{and} \qquad \psi_i = \frac{Q_i(1, r_i; r_i, 1)}{\nu_i B(1, r_i)P_i(1, r_i; r_i, 1)}.$$

As a consequence, $K(\mathrm{Leg}\mathscr{H})$ is holomorphic on $D_\ell = \{\check{y} = 0\}$ if and only if

$$\left(1 + \frac{1}{\nu_0}\right)\frac{(p_0 - r_0)Q_0(1, r_0; -r_0, 1)}{B(1, r_0)P_0(1, r_0; -r_0, 1)} + \sum_{i=1}^{n}\left(1 - \frac{1}{\nu_i}\right)(p_0 - r_i)$$
$$\times\left(\frac{Q_i(1, r_i; r_i, 1)}{B(1, r_i)P_i(1, r_i; r_i, 1)} + \frac{3\nu_i(p_0 - r_0)}{r_i - r_0}\right) = 0.$$

This ends the proof of the theorem. □

Corollary 3.20 *Let $\mathscr{H} = \ell \boxtimes \mathcal{H}$ be a homogeneous pre-foliation of co-degree 1 and degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$, defined by the 1-form*

$$\omega = (\alpha x + \beta y)\left(A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y\right), \quad A, B \in \mathbb{C}[x, y]_{d-1}, \quad \gcd(A, B) = 1.$$

Assume that the line ℓ is not invariant by $\mathcal{H}$ and that the fiber $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}\big(\underline{\mathcal{G}}_{\mathcal{H}}([-\alpha:\beta])\big)$ does not contain any non-fixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$. Then the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic on $D_\ell = \mathcal{G}_{\mathcal{H}}(\ell)$ if and only if $Q(\beta,-\alpha;\alpha,\beta)=0$, where

$$Q(x,y;\alpha,\beta) := \begin{vmatrix} \dfrac{\partial P}{\partial x} & A(\beta,-\alpha) \\ \dfrac{\partial P}{\partial y} & B(\beta,-\alpha) \end{vmatrix}$$

and

$$P(x,y;\alpha,\beta) := \frac{\begin{vmatrix} A(x,y) & A(\beta,-\alpha) \\ B(x,y) & B(\beta,-\alpha) \end{vmatrix}}{\alpha x+\beta y}. \tag{3.6}$$

Remark 3.21 In particular, in degree $d=3$, the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic along D_ℓ if and only if the line with equation $A(\beta,-\alpha)x + B(\beta,-\alpha)y = 0$ is invariant by $\mathcal{H}$, or equivalently, if and only if $\underline{\mathcal{G}}_{\mathcal{H}}\big(\underline{\mathcal{G}}_{\mathcal{H}}([-\alpha:\beta])\big) = \underline{\mathcal{G}}_{\mathcal{H}}([-\alpha:\beta])$.

Indeed, putting $a = A(\beta,-\alpha)$, $b = B(\beta,-\alpha)$ and $P(x,y;\alpha,\beta) = f(\alpha,\beta)x + g(\alpha,\beta)y$ we obtain

$$\begin{aligned} Q(\beta,-\alpha;\alpha,\beta) &= f(\alpha,\beta)b - g(\alpha,\beta)a = P(b,-a;\alpha,\beta) \\ &= -\frac{bA(b,-a)-aB(b,-a)}{\beta a-\alpha b} = -\frac{\mathrm{C}_{\mathcal{H}}(b,-a)}{\mathrm{C}_{\mathcal{H}}(\beta,-\alpha)}, \end{aligned}$$

where $\mathrm{C}_{\mathcal{H}} = xA + yB$ denotes the tangent cone of $\mathcal{H}$ at the origin O, *see* [4, Section 2].

Combining Corollaries 3.12 and 3.20, we obtain:

Corollary 3.22 *Let $\mathscr{H} = \ell \boxtimes \mathcal{H}$ be a homogeneous pre-foliation of co-degree 1 and degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$, defined by the 1-form*

$$\omega = (\alpha x+\beta y)\,(A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y)\,,\quad A,B\in\mathbb{C}[x,y]_{d-1},\ \gcd(A,B)=1.$$

Assume that the homogeneous foliation $\mathcal{H}$ is convex and that the line ℓ is not invariant by $\mathcal{H}$. Then the d-web $\mathrm{Leg}\mathscr{H}$ is flat if and only if $Q(\beta,-\alpha;\alpha,\beta)=0$, where Q is the polynomial given by (3.6).

Here are two examples that will be useful in Section §5.

Example 3.23 Let us consider the homogeneous foliation $\mathcal{H}_0^{d-1}$ defined in the affine chart $z=1$ by the 1-form

$$\omega_0^{d-1} = (d-2)y^{d-1}\mathrm{d}x + x\left(x^{d-2}-(d-1)y^{d-2}\right)\mathrm{d}y.$$

We know from [4, Example 6.5] that $\mathcal{H}_0^{d-1}$ is convex, of type $1\cdot\mathrm{R}_{d-2} + (d-2)\cdot\mathrm{R}_1$ and with inflection divisor

$$\mathrm{I}_{\mathcal{H}_0^{d-1}} = \mathrm{I}^{\mathrm{inv}}_{\mathcal{H}_0^{d-1}} = -(d-1)(d-2)xzy^{d-1}\big(y^{d-2}-x^{d-2}\big)^2.$$

If ℓ is one of the invariant lines of $\mathcal{H}_0^{d-1}$, *i.e.* if $\ell \in \{xyz(y-\zeta^k x)=0,\ k=0,\ldots,d-3\}$, where $\zeta = \exp\left(\frac{2i\pi}{d-2}\right)$, then the d-web $\mathrm{Leg}(\ell\boxtimes\mathcal{H}_0^{d-1})$ is flat by Corollary 3.11.

If $\ell = (y - \rho x = 0)$ is not invariant by $\mathcal{H}_0^{d-1}$, then the d-web $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_0^{d-1})$ is flat if and only if $\rho^{d-2} = \frac{1}{2(d-2)}$, *i.e.* if and only if $\ell \in \left\{y - \rho_0 \zeta^k x = 0,\ k = 0, \ldots, d-3\right\}$, where $\rho_0 = \sqrt[d-2]{\frac{1}{2d-4}}$. Indeed, with the notations of Corollary 3.20, we have

$$Q(x, y; -\rho, 1) = \left(1 - (d-1)\rho^{d-2}\right) \frac{\partial P}{\partial x} - (d-2)\rho^{d-1} \frac{\partial P}{\partial y}$$

and

$$\begin{aligned} P(x, y; -\rho, 1) &= -(d-2)\left((d-1)(\rho y)^{d-2} - \frac{y^{d-1} - (\rho x)^{d-1}}{y - \rho x}\right) \\ &= -(d-2)\left((d-1)(\rho y)^{d-2} - \sum_{i=0}^{d-2} \rho^i x^i y^{d-2-i}\right), \end{aligned}$$

so that, according to Corollary 3.22, the flatness of $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_0^{d-1})$ is characterized by

$$0 = Q(1, \rho; -\rho, 1) = \frac{1}{2}(d-1)(d-2)^2\rho^{d-2}\left(\rho^{d-2} - 1\right)\left((2d-4)\rho^{d-2} - 1\right) \Longleftrightarrow \rho^{d-2} = \frac{1}{2(d-2)}.$$

In all cases, for any line $\ell \subset \mathbb{P}^2_{\mathbb{C}}$ such that $O \in \ell$ or $\ell = L_\infty$, the d-web $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_0^{d-1})$ is flat if and only if, up to linear conjugation, $\ell = L_\infty$ or $\ell \in \{xy(y-x)(y - \rho_0 x) = 0\}$. Indeed, putting $\varphi(x, y) = (x, \zeta^k y)$, we have

$$\varphi^*\left(\left(y - \zeta^k x\right)\omega_0^{d-1}\right) = \zeta^{2k}\left(y - x\right)\omega_0^{d-1}$$

and

$$\varphi^*\left(\left(y - \rho_0\zeta^k x\right)\omega_0^{d-1}\right) = \zeta^{2k}\left(y - \rho_0 x\right)\omega_0^{d-1}.$$

Example 3.24 For $d \geq 4$, let $\mathcal{H}_4^{d-1}$ be the homogeneous foliation defined in the affine chart $z = 1$ by the 1-form

$$\omega_4^{d-1} = y(\sigma_d x^{d-2} - y^{d-2})\mathrm{d}x + x(\sigma_d y^{d-2} - x^{d-2})\mathrm{d}y, \qquad \text{where } \sigma_d = 1 + \tfrac{2}{d-3}.$$

This foliation is convex of type $(d-2) \cdot \mathrm{R}_2$; indeed, a straightforward computation shows that

$$\mathrm{I}_{\mathcal{H}_4^{d-1}} = \mathrm{I}^{\mathrm{inv}}_{\mathcal{H}_4^{d-1}} = \sigma_d(\sigma_d - 1)xyz\left(x^{d-2} + y^{d-2}\right)^3.$$

Let ℓ be a line of $\mathbb{P}^2_{\mathbb{C}}$ such that $O \in \ell$ or $\ell = L_\infty$. If ℓ is invariant by $\mathcal{H}_4^{d-1}$, then Corollary 3.11 ensures that $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_4^{d-1})$ is flat, and we have $\ell \in \{xyz(y - \xi^{2k+1} x) = 0,\ k = 0, \ldots, d-3\}$, where $\xi = \exp\left(\frac{\mathrm{i}\pi}{d-2}\right)$.

If ℓ is not invariant by $\mathcal{H}_4^{d-1}$, then $\ell = \{y - \rho x = 0\}$ with $\rho(\rho^{d-2} + 1) \neq 0$; by applying Corollary 3.22, we obtain that the d-web $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_4^{d-1})$ is flat if and only if

$$0 = Q(1, \rho; -\rho, 1) = -\sigma_d(d-2)(\rho^{d-2} + 1)^2(\rho^{d-2} - 1),$$

hence if and only if $\rho^{d-2} = 1$, which is equivalent to $\ell \in \{y - \xi^{2k} x = 0,\ k = 0, \ldots, d-3\}$.

Note that, in all cases, the d-web $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_4^{d-1})$ is flat if and only if, up to linear conjugation, $\ell = L_\infty$ or $\ell \in \{x(y-x)(y - \xi x) = 0\}$. Indeed, putting $\varphi(x, y) = (y, x)$ and $\psi(x, y) = (x, \xi^{2k} y)$, we have

$$\varphi^*(y\omega_4^{d-1}) = x\omega_4^{d-1},\ \psi^*\left(\left(y - \xi^{2k} x\right)\omega_4^{d-1}\right) = \xi^{4k}\left(y - x\right)\omega_4^{d-1},$$

$$\psi^*\left(\left(y-\xi^{2k+1}x\right)\omega_4^{d-1}\right)=\xi^{4k}\left(y-\xi x\right)\omega_4^{d-1}.$$

Corollary 3.25 *Let $d\geq 3$ be an integer and let $\mathcal{H}$ be a homogeneous foliation of degree $d-1$ on $\mathbb{P}^2_{\mathbb{C}}$ defined by the 1-form*

$$\omega=A(x,y)\mathrm{d}x+B(x,y)\mathrm{d}y,\quad A,B\in\mathbb{C}[x,y]_{d-1},\quad \gcd(A,B)=1.$$

Assume that $\mathcal{H}$ admits a transverse inflection line $\ell=(\alpha\,x+\beta\,y=0)$ of order $\nu-1$. Assume moreover that $[-\alpha:\beta]\in\mathbb{P}^1_{\mathbb{C}}$ is the only non-fixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$ in its fiber $\underline{\mathcal{G}}_{\mathcal{H}}^{-1}(\underline{\mathcal{G}}_{\mathcal{H}}([-\alpha:\beta]))$. Put $\mathscr{H}:=\ell\boxtimes\mathcal{H}$. Then the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic along D_ℓ if and only if $Q(\beta,-\alpha;\alpha,\beta)=0$, where

$$Q(x,y;\alpha,\beta):=(\nu-1)\left(\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}\right)P(x,y;\alpha,\beta)+(2\nu+1)\begin{vmatrix}\frac{\partial P}{\partial x} & A(x,y)\\ \frac{\partial P}{\partial y} & B(x,y)\end{vmatrix}\quad \text{and}$$

$$P(x,y;\alpha,\beta):=\frac{\begin{vmatrix}A(x,y) & A(\beta,-\alpha)\\ B(x,y) & B(\beta,-\alpha)\end{vmatrix}}{(\alpha\,x+\beta\,y)^{\nu}}.$$

Corollary 3.26 *Let $d\geq 3$ be an integer and let $\mathcal{H}$ be a homogeneous foliation of degree $d-1$ on $\mathbb{P}^2_{\mathbb{C}}$ defined by the 1-form*

$$\omega=A(x,y)\mathrm{d}x+B(x,y)\mathrm{d}y,\quad A,B\in\mathbb{C}[x,y]_{d-1},\quad \gcd(A,B)=1.$$

Assume that $\mathcal{H}$ has a transverse inflection line $\ell=(\alpha\,x+\beta\,y=0)$ of maximal order $d-2$. Put $\mathscr{H}:=\ell\boxtimes\mathcal{H}$. Then the curvature of $\mathrm{Leg}\mathscr{H}$ is holomorphic along D_ℓ if and only if the 2-form $\mathrm{d}\omega$ vanishes on the line ℓ.

Remark 3.27 When $d\geq 4$ the condition «$\mathrm{d}\omega$ vanishes on the line ℓ» also expresses the holomorphy of the curvature of $\mathrm{Leg}\mathcal{H}$ along D_ℓ, thanks to [4, Theorem 3.8]. Thus Corollary 3.26 establishes the equivalence between the holomorphy on D_ℓ of $K(\mathrm{Leg}\mathscr{H})$ and that of $K(\mathrm{Leg}\mathcal{H})$.

4 Flatness and homogeneous pre-foliations $\ell\boxtimes\mathcal{H}$ of co-degree 1 such that $\deg\mathcal{T}_{\mathcal{H}}=2$

In this section we propose to classify, up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$, all homogeneous pre-foliations $\mathscr{H}=\ell\boxtimes\mathcal{H}$ of co-degree 1 and degree $d\geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$ such that $\deg\mathcal{T}_{\mathcal{H}}=2$ and the d-web $\mathrm{Leg}\mathscr{H}$ is flat. The equality $\deg\mathcal{T}_{\mathcal{H}}=2$ holds if and only if the type $\mathcal{T}_{\mathcal{H}}$ of $\mathcal{H}$ is of one of the following three forms: $2\cdot\mathrm{R}_{d-2}$, $2\cdot\mathrm{T}_{d-2}$, $1\cdot\mathrm{R}_{d-2}+1\cdot\mathrm{T}_{d-2}$. According to [4, Proposition 4.1], every homogeneous foliation of type $2\cdot\mathrm{R}_{d-2}$ is linearly conjugate to the convex foliation $\mathcal{H}_1^{d-1}$ defined by the 1-form

$$\omega_1^{d-1}=y^{d-1}\mathrm{d}x-x^{d-1}\mathrm{d}y.$$

The homogeneous foliations of type $2\cdot\mathrm{T}_{d-2}$, resp. $1\cdot\mathrm{R}_{d-2}+1\cdot\mathrm{T}_{d-2}$, are given, up to linear conjugation, by

$$\omega_2^{d-1}(\lambda,\mu)=(x^{d-1}+\lambda\,y^{d-1})\mathrm{d}x+(\mu\,x^{d-1}-y^{d-1})\mathrm{d}y,\quad \text{where } \lambda,\mu\in\mathbb{C}, \text{ with } \lambda\mu\neq -1,$$

resp. $\omega_3^{d-1}(\lambda)=(x^{d-1}+\lambda y^{d-1})\mathrm{d}x+x^{d-1}\mathrm{d}y,\quad \text{where } \lambda\in\mathbb{C}^*,$

cf. proof of [4, Proposition 4.1]. We will denote by $\mathcal{H}_2^{d-1}(\lambda, \mu)$, resp. $\mathcal{H}_3^{d-1}(\lambda)$, the foliation defined by $\omega_2^{d-1}(\lambda, \mu)$, resp. $\omega_3^{d-1}(\lambda)$.

In the following three lemmas, ℓ denotes a line of $\mathbb{P}^2_{\mathbb{C}}$ such that $O \in \ell$ or $\ell = L_\infty$.

Lemma 4.1 *The d-web $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_1^{d-1})$ is flat if and only if, up to linear conjugation, $\ell = L_\infty$ or $\ell \in \{x(y-x)(y-\xi\, x) = 0\}$, where $\xi = \exp\left(\frac{\mathrm{i}\pi}{d-2}\right)$.*

Proof Note first of all that the foliation $\mathcal{H}_1^{d-1}$ has inflection divisor

$$\mathrm{I}_{\mathcal{H}_1^{d-1}} = \mathrm{I}^{\mathrm{inv}}_{\mathcal{H}_1^{d-1}} = (d-1)z\, x^{d-1} y^{d-1}\big(y^{d-2} - x^{d-2}\big).$$

i. If ℓ is invariant by $\mathcal{H}_1^{d-1}$, then $\ell \in \{xyz(y - \xi^{2k}x) = 0,\ k = 0, \ldots, d-3\}$ and the d-web $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_1^{d-1})$ is flat (Corollary 3.11).

ii. Assume that ℓ is not invariant by $\mathcal{H}_1^{d-1}$; then $\ell = (y - \rho x = 0)$ with $\rho(\rho^{d-2} - 1) \neq 0$. According to Corollary 3.22, the d-web $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_1^{d-1})$ is flat if and only if $Q(1, \rho; -\rho, 1) = 0$, where

$$Q(x, y; -\rho, 1) = -\frac{\partial P}{\partial x} - \rho^{d-1}\frac{\partial P}{\partial y} \qquad \text{and}$$

$$P(x, y; -\rho, 1) = -\frac{y^{d-1} - (\rho x)^{d-1}}{y - \rho x} = -\sum_{i=0}^{d-2} \rho^i x^i y^{d-2-i}.$$

Thus $Q(1, \rho; -\rho, 1) = \frac{1}{2}(d-1)(d-2)\rho^{d-2}(\rho^{d-2}+1)$, and the flatness of $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_1^{d-1})$ is equivalent to $\rho^{d-2} = -1$ and therefore to $\ell \in \{y - \xi^{2k+1}x = 0,\ k = 0, \ldots, d-3\}$.

In the two cases considered, $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_1^{d-1})$ is flat if and only if, up to conjugation, $\ell = L_\infty := (z = 0)$ or $\ell \in \{x(y-x)(y-\xi\, x) = 0\}$. Indeed, putting $\varphi_1(x, y) = (y, x)$ and $\varphi_2(x, y) = (x, \xi^{2k} y)$, we have

$$\varphi_1^*(y\omega_1^{d-1}) = -x\omega_1^{d-1}, \qquad \varphi_2^*\left((y - \xi^{2k}x)\omega_1^{d-1}\right) = \xi^{4k}(y-x)\omega_1^{d-1},$$

$$\varphi_2^*\left((y - \xi^{2k+1}x)\omega_1^{d-1}\right) = \xi^{4k}(y - \xi\, x)\omega_1^{d-1}.$$

□

Lemma 4.2 *The d-web $\mathrm{Leg}\big(\ell \boxtimes \mathcal{H}_2^{d-1}(\lambda, \mu)\big)$ is flat if and only if, up to linear conjugation, one of the following cases occurs:*

(i) $\ell = L_\infty$ *and* $d = 3$;
(ii) $\ell = L_\infty$, $d \geq 4$ *and* $\lambda = \mu = 0$;
(iii) $\ell = (x = 0)$ *and* $\lambda = \mu = 0$;
(iv) $\ell = (y - x = 0)$, $d \geq 4$ *and* $(\lambda, \mu) = \left(\frac{3}{d}, -\frac{3}{d}\right)$;
(v) $\ell = (y - \xi' x = 0)$, $d \geq 4$ *and* $(\lambda, \mu) = \left(\frac{3\xi'}{d}, -\frac{3}{d\xi'}\right)$, *where* $\xi' = \exp\left(\frac{\mathrm{i}\pi}{d}\right)$.

Proof We have $\omega_2^{d-1}(\lambda, \mu) = A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y$, where $A(x, y) = x^{d-1} + \lambda\, y^{d-1}$ and $B(x, y) = \mu\, x^{d-1} - y^{d-1}$; an immediate computation shows that

$$\mathrm{I}^{\mathrm{inv}}_{\mathcal{H}_2^{d-1}(\lambda,\mu)} = z(x^d + \mu\, x^{d-1}y + \lambda\, xy^{d-1} - y^d)$$

and

$$\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}_2^{d-1}(\lambda,\mu)} = x^{d-2}y^{d-2}.$$

1. If $\ell = L_\infty$ and $d = 3$, then the web $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_2^2(\lambda, \mu))$ is flat by Corollary 3.2.

2. Assume that $\ell = L_\infty$ and $d \geq 4$. Then, according to [4, Theorem 3.1 and 3.8], the web $\mathrm{Leg}(\mathcal{H}_2^{d-1}(\lambda, \mu))$ is flat if and only if $\mathrm{d}\big(\omega_2^{d-1}(\lambda, \mu)\big)$ vanishes on the two lines $xy = 0$. Now,

$$\mathrm{d}\big(\omega_2^{d-1}(\lambda, \mu)\big)\Big|_{x=0} = -(d-1)\lambda\, y^{d-2}\mathrm{d}x \wedge \mathrm{d}y$$

and

$$\mathrm{d}\big(\omega_2^{d-1}(\lambda, \mu)\big)\Big|_{y=0} = (d-1)\mu\, x^{d-2}\mathrm{d}x \wedge \mathrm{d}y.$$

Therefore $\mathrm{Leg}(\mathcal{H}_2^{d-1}(\lambda, \mu))$ is flat if and only if $\lambda = \mu = 0$; hence the same holds for $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_2^{d-1}(\lambda, \mu))$ (Theorem 3.1).

3. Let us consider the case where $\ell \in \{xy = 0\}$. Up to permuting the coordinates x and y, we can assume that $\ell = (x = 0)$. According to Theorem 3.7, the d-web $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_2^{d-1}(\lambda, \mu))$ is flat if and only if its curvature is holomorphic on $\mathcal{G}_{\mathcal{H}_2^{d-1}(\lambda,\mu)}(\{xy = 0\})$. Now, on the one hand, $K(\mathrm{Leg}(\ell \boxtimes \mathcal{H}_2^{d-1}(\lambda, \mu)))$ is holomorphic on $D_\ell = \mathcal{G}_{\mathcal{H}_2^{d-1}(\lambda,\mu)}(\ell)$ if and only if $\mathrm{d}\big(\omega_2^{d-1}(\lambda, \mu)\big)$ vanishes on $\ell = (x = 0)$ (Corollary 3.26), *i.e.* if and only if $\lambda = 0$. On the other hand, according to Corollary 3.17, $K(\mathrm{Leg}(\ell \boxtimes \mathcal{H}_2^{d-1}(\lambda, \mu)))$ is holomorphic on $\mathcal{G}_{\mathcal{H}_2^{d-1}(\lambda,\mu)}(\{y = 0\})$ if and only if

$$0 = (d-3)\big(\partial_x B(1,0) - \partial_y A(1,0)\big) + 3(d-1)B(1,0) = d(d-1)\mu \Longleftrightarrow \mu = 0.$$

It follows that $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_2^{d-1}(\lambda, \mu))$ is flat if and only if $\lambda = \mu = 0$.

4. Let us examine the case where $\ell = (y - \rho x = 0)$ with $\rho \neq 0$. By Corollary 3.17, $K(\mathrm{Leg}(\ell \boxtimes \mathcal{H}_2^{d-1}(\lambda, \mu)))$ is holomorphic on $\mathcal{G}_{\mathcal{H}_2^{d-1}(\lambda,\mu)}(\{xy = 0\})$ if and only if

$$\begin{cases} 0 = -(d-3)\Big(\partial_x B(0,-1) - \partial_y A(0,-1)\Big) - 3(d-1)\Big(A(0,-1) + \rho B(0,-1)\Big) \\ \quad = (-1)^d (d-1)(d\lambda - 3\rho) \\ 0 = -\rho(d-3)\Big(\partial_x B(1,0) - \partial_y A(1,0)\Big) - 3(d-1)\Big(A(1,0) + \rho B(1,0)\Big) = -(d-1)(d\rho\mu + 3), \end{cases}$$

i.e. if and only if $\lambda = \lambda_0 := \frac{3\rho}{d}$, $\mu = \mu_0 := -\frac{3}{d\rho}$ and $d \neq 3$, because $\lambda\mu \neq -1$. We now distinguish two cases according to whether or not ℓ is invariant by $\mathcal{H}_2^{d-1}(\lambda_0, \mu_0)$.

4.1. Assume that ℓ is invariant by $\mathcal{H}_2^{d-1}(\lambda_0, \mu_0)$. Then the dual web of $\ell \boxtimes \mathcal{H}_2^{d-1}(\lambda_0, \mu_0)$ is flat by Theorem 3.7. Since $\mathrm{I}^{\mathrm{inv}}_{\mathcal{H}_2^{d-1}(\lambda_0,\mu_0)}\Big|_{y=\rho x} = \left(\frac{3}{d} - 1\right)\left(\rho^d - 1\right) z\, x^d$, the invariance of ℓ by $\mathcal{H}_2^{d-1}(\lambda_0, \mu_0)$ is equivalent to $\rho^d = 1$; as a consequence $(\rho, \lambda_0, \mu_0) \in \left\{\left(\xi'^{2k}, \frac{3\xi'^{2k}}{d}, -\frac{3}{d\xi'^{2k}}\right), k = 0, \ldots, d-1\right\}$. Up to conjugation, $(\rho, \lambda_0, \mu_0) = (1, \frac{3}{d}, -\frac{3}{d})$; indeed, putting $\varphi(x, y) = (x, \xi'^{2k} y)$ we have

$$\varphi^*\left(\left(y - \xi'^{2k}x\right)\omega_2^{d-1}\left(\frac{3\xi'^{2k}}{d}, -\frac{3}{d\xi'^{2k}}\right)\right) = \xi'^{2k}\,(y - x)\,\omega_2^{d-1}\left(\tfrac{3}{d}, -\tfrac{3}{d}\right).$$

4.2. Assume that ℓ is not invariant by $\mathcal{H}_2^{d-1}(\lambda_0, \mu_0)$. Then, by Theorem 3.7 and Corollary 3.20, the flatness of $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_2^{d-1}(\lambda_0, \mu_0))$ translates into $Q(1, \rho; -\rho, 1) = 0$, where

$$Q(x, y; -\rho, 1) = \left(\mu_0 - \rho^{d-1}\right)\frac{\partial P}{\partial x} - \left(\lambda_0 \rho^{d-1} + 1\right)\frac{\partial P}{\partial y}$$

and

$$P(x, y; -\rho, 1) = \frac{(\lambda_0\mu_0 + 1)\left(y^{d-1} - (\rho x)^{d-1}\right)}{y - \rho x} = (\lambda_0\mu_0 + 1)\sum_{i=0}^{d-2} \rho^i x^i y^{d-2-i}.$$

Hence $Q(1, \rho; -\rho, 1) = \frac{1}{2}\left(\frac{3}{d} - 1\right)\left(\frac{3}{d} + 1\right)^2 (d-1)(d-2)\rho^{d-3}(\rho^d + 1)$, and consequently $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_2^{d-1}(\lambda_0, \mu_0))$ is flat if and only if $\rho^d = -1$, hence if and only if $(\rho, \lambda_0, \mu_0) \in \left\{\left(\xi'^{2k+1}, \frac{3\xi'^{2k+1}}{d}, -\frac{3}{d\xi'^{2k+1}}\right), k = 0, \ldots, d-1\right\}$, or equivalently, if and only if, up to conjugation, $(\rho, \lambda_0, \mu_0) = \left(\xi', \frac{3\xi'}{d}, -\frac{3}{d\xi'}\right)$, because

$$\varphi^*\left((y - \xi'^{2k+1}x)\omega_2^{d-1}\left(\frac{3\xi'^{2k+1}}{d}, -\frac{3}{d\xi'^{2k+1}}\right)\right) = \xi'^{2k}\left(y - \xi' x\right)\omega_2^{d-1}\left(\frac{3\xi'}{d}, -\frac{3}{d\xi'}\right).$$

□

Lemma 4.3 *The d-web $\mathrm{Leg}\big(\ell \boxtimes \mathcal{H}_3^{d-1}(\lambda)\big)$ is flat if and only if one of the following cases holds:*

(i) $\ell = L_\infty$ *and* $d = 3$;

(ii) $\ell = (dy + 3x = 0)$, $d \geq 4$ *and* $\lambda = \dfrac{(-1)^d(d-3)d^{d-2}}{3^{d-1}}$;

(iii) $\ell = (dy + 3x = 0)$ *and* $\lambda = \dfrac{(-1)^d(d+3)d^{d-2}}{3^{d-1}}$.

Proof We have $\omega_3^{d-1}(\lambda) = A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y$, where $A(x, y) = x^{d-1} + \lambda y^{d-1}$ and $B(x, y) = x^{d-1}$; an immediate computation leads to

$$\mathrm{I}^{\mathrm{inv}}_{\mathcal{H}_3^{d-1}(\lambda)} = z\,x^{d-1}\big(x^{d-1} + x^{d-2}y + \lambda\,y^{d-1}\big)$$

and

$$\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}_3^{d-1}(\lambda)} = y^{d-2}.$$

1. Assume that $\ell = L_\infty$. If $d = 3$, then the web $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_3^2(\lambda))$ is flat, thanks to Corollary 3.2. For $d \geq 4$, the webs $\mathrm{Leg}(\mathcal{H}_3^{d-1}(\lambda))$ and $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_3^{d-1}(\lambda))$ have the same curvature (Theorem 3.1) and cannot be flat. Indeed, we have

$$\mathrm{d}\big(\omega_3^{d-1}(\lambda)\big)\Big|_{y=0} = (d-1)x^{d-2}\mathrm{d}x \wedge \mathrm{d}y \not\equiv 0;$$

this implies, according to [4, Theorem 3.8], that $K(\mathrm{Leg}(\mathcal{H}_3^{d-1}(\lambda)))$ cannot be holomorphic along $\mathcal{G}_{\mathcal{H}_3^{d-1}(\lambda)}(\{y = 0\})$.

2. If $\ell = (y = 0)$, then the fact that $\mathrm{d}\big(\omega_3^{d-1}(\lambda)\big)$ does not vanish on ℓ implies, by Corollary 3.26, that $K(\mathrm{Leg}(\ell \boxtimes \mathcal{H}_3^{d-1}(\lambda)))$ cannot be holomorphic on $\mathcal{G}_{\mathcal{H}_3^{d-1}(\lambda)}(\ell)$, so that $\mathrm{Leg}(\ell \boxtimes \mathcal{H}_3^{d-1}(\lambda))$ cannot be flat.

3. Assume that $\ell = (x - \rho\,y = 0)$, where $\rho \in \mathbb{C}$. By Corollary 3.17, $K(\mathrm{Leg}(\ell\boxtimes\mathcal{H}_3^{d-1}(\lambda)))$ is holomorphic on $\mathcal{G}_{\mathcal{H}_3^{d-1}(\lambda)}(\{y = 0\})$ if and only if

$$0 = (d-3)\Big(\partial_x B(1, 0) - \partial_y A(1, 0)\Big) + 3(d-1)\Big(B(1, 0) + \rho A(1, 0)\Big) = (d-1)(3\rho + d),$$

hence if and only if $\rho = -\frac{d}{3}$, which is equivalent to $\ell = \ell_0$ where $\ell_0 = (dy + 3x = 0)$. Then we have to distinguish two cases:

3.1. If ℓ_0 is invariant by $\mathcal{H}_3^{d-1}(\lambda)$, then Theorem 3.7 ensures that the d-web $\mathrm{Leg}(\ell_0 \boxtimes \mathcal{H}_3^{d-1}(\lambda))$ is flat; since

$$\mathrm{I}^{\mathrm{inv}}_{\mathcal{H}_3^{d-1}(\lambda)}\Big|_{x=-\frac{d}{3}y} = -d^{d-1}z\left(\frac{y}{3}\right)^{2d-2}\left((-1)^d 3^{d-1}\lambda - (d-3)d^{d-2}\right),$$

the invariance of ℓ_0 by $\mathcal{H}_3^{d-1}(\lambda)$ is characterized by $\lambda = \dfrac{(-1)^d(d-3)d^{d-2}}{3^{d-1}}$ and $d \neq 3$, because $\lambda \neq 0$.

3.2. Assume that ℓ_0 is not invariant by $\mathcal{H}_3^{d-1}(\lambda)$. Then, according to Theorem 3.7 and Corollary 3.20, the d-web $\mathrm{Leg}(\ell_0 \boxtimes \mathcal{H}_3^{d-1}(\lambda))$ is flat if and only if $Q(d,-3;3,d) = 0$, where

$$Q(x,y;3,d) = d^{d-1}\frac{\partial P}{\partial x} - \left(d^{d-1} + (-3)^{d-1}\lambda\right)\frac{\partial P}{\partial y}$$

and

$$P(x,y;3,d) = \frac{\lambda\left((dy)^{d-1} - (-3x)^{d-1}\right)}{dy+3x} = \lambda\sum_{i=0}^{d-2}(-3x)^i(dy)^{d-2-i}.$$

Thus $Q(d,-3;3,d) = -\frac{1}{6}\lambda(d-1)(d-2)(3d)^{d-2}\left(3^{d-1}\lambda - (-1)^d(d+3)d^{d-2}\right)$ and the flatness of $\mathrm{Leg}(\ell_0 \boxtimes \mathcal{H}_3^{d-1}(\lambda))$ translates into $\lambda = \dfrac{(-1)^d(d+3)d^{d-2}}{3^{d-1}}$. □

Lemmas 4.1, 4.2 and 4.3 imply the following proposition.

Proposition 4.4 *Let $\mathscr{H} = \ell \boxtimes \mathcal{H}$ be a homogeneous pre-foliation of co-degree 1 and degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$. Assume that $\deg \mathcal{T}_{\mathcal{H}} = 2$, or equivalently that the map $\mathcal{G}_{\mathcal{H}}$ has exactly two critical points. Then, for $d \geq 4$ the web $\mathrm{Leg}\mathscr{H}$ is flat if and only if $\mathscr{H}$ is linearly conjugate to one of the ten following pre-foliations*

1. $\mathscr{H}_1^d = L_\infty \boxtimes \mathcal{H}_1^{d-1}$;
2. $\mathscr{H}_2^d = \{x=0\} \boxtimes \mathcal{H}_1^{d-1}$;
3. $\mathscr{H}_3^d = \{y-x=0\} \boxtimes \mathcal{H}_1^{d-1}$;
4. $\mathscr{H}_4^d = \{y-\xi x=0\} \boxtimes \mathcal{H}_1^{d-1}$, *where* $\xi = \exp\left(\frac{i\pi}{d-2}\right)$;
5. $\mathscr{H}_5^d = \{x=0\} \boxtimes \mathcal{H}_2^{d-1}(0,0)$;
6. $\mathscr{H}_6^d = \{dy+3x=0\} \boxtimes \mathcal{H}_3^{d-1}(\lambda_0)$, *where* $\lambda_0 = \frac{(-1)^d(d+3)d^{d-2}}{3^{d-1}}$;
7. $\mathscr{H}_7^d = \{dy+3x=0\} \boxtimes \mathcal{H}_3^{d-1}(\lambda_1)$, *where* $\lambda_1 = \frac{(-1)^d(d-3)d^{d-2}}{3^{d-1}}$;
8. $\mathscr{H}_8^d = L_\infty \boxtimes \mathcal{H}_2^{d-1}(0,0)$;
9. $\mathscr{H}_9^d = \{y-x=0\} \boxtimes \mathcal{H}_2^{d-1}\left(\frac{3}{d},-\frac{3}{d}\right)$;
10. $\mathscr{H}_{10}^d = \{y-\xi' x=0\} \boxtimes \mathcal{H}_2^{d-1}\left(\frac{3\xi'}{d},-\frac{3}{d\xi'}\right)$, *where* $\xi' = \exp\left(\frac{i\pi}{d}\right)$.

For $d=3$ the web $\mathrm{Leg}\mathscr{H}$ is flat if and only if, up to linear conjugation, either $\mathscr{H}$ is one of the six pre-foliations $\mathscr{H}_1^3, \mathscr{H}_2^3, \ldots, \mathscr{H}_6^3$, or $\mathscr{H}$ is of one of the following two types

11. $\mathscr{H}_7^3(\lambda) = L_\infty \boxtimes \mathcal{H}_3^2(\lambda)$, *where* $\lambda \in \mathbb{C}^*$;
12. $\mathscr{H}_8^3(\lambda,\mu) = L_\infty \boxtimes \mathcal{H}_2^2(\lambda,\mu)$, *where* $\lambda,\mu \in \mathbb{C}$ *with* $\lambda\mu \neq -1$.

Combining Proposition 4.4 with the fact that every homogeneous foliation of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ has degree of type 2, we obtain the classification, up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$, of homogeneous pre-foliations of type $(1,3)$ on $\mathbb{P}^2_{\mathbb{C}}$ whose dual web is flat.

Table 1 Types and CAMACHO- SAD polynomials of the foliations $\mathcal{H}_1^2$, $\mathcal{H}_2^2(0,0)$ and $\mathcal{H}_3^2(-2)$

$\mathcal{H}$	$\mathcal{T}_{\mathcal{H}}$	$\mathrm{CS}_{\mathcal{H}}(\lambda)$
$\mathcal{H}_1^2$	$2 \cdot \mathrm{R}_1$	$(\lambda-1)^2(\lambda+1)$
$\mathcal{H}_2^2(0,0)$	$2 \cdot \mathrm{T}_1$	$(\lambda-\frac{1}{3})^3$
$\mathcal{H}_3^2(-2)$	$1 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{T}_1$	$(\lambda-1)(\lambda-\frac{1}{3})(\lambda+\frac{1}{3})$

Corollary 4.5 *Up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$, there are six examples and two families of homogeneous pre-foliations of co-degree 1 and degree 3 on $\mathbb{P}^2_{\mathbb{C}}$ with a flat* LEGENDRE *transform, namely:*

1. $\mathscr{H}_1^3 = L_\infty \boxtimes \mathcal{H}_1^2$;
2. $\mathscr{H}_2^3 = \{x=0\} \boxtimes \mathcal{H}_1^2$;
3. $\mathscr{H}_3^3 = \{y-x=0\} \boxtimes \mathcal{H}_1^2$;
4. $\mathscr{H}_4^3 = \{y+x=0\} \boxtimes \mathcal{H}_1^2$;
5. $\mathscr{H}_5^3 = \{x=0\} \boxtimes \mathcal{H}_2^2(0,0)$;
6. $\mathscr{H}_6^3 = \{y+x=0\} \boxtimes \mathcal{H}_3^2(-2)$;
7. $\mathscr{H}_7^3(\lambda) = L_\infty \boxtimes \mathcal{H}_3^2(\lambda)$, *where* $\lambda \in \mathbb{C}^*$;
8. $\mathscr{H}_8^3(\lambda,\mu) = L_\infty \boxtimes \mathcal{H}_2^2(\lambda,\mu)$, *where* $\lambda, \mu \in \mathbb{C}$ *with* $\lambda\mu \neq -1$.

In Section §6 we will need, for $\mathcal{H} \in \{\mathcal{H}_1^2, \mathcal{H}_2^2(0,0), \mathcal{H}_3^2(-2)\}$, the values of the CAMACHO-SAD indices $\mathrm{CS}(\mathcal{H}, L_\infty, s)$, $s \in \mathrm{Sing}\mathcal{H} \cap L_\infty$. For this, we have computed, for each of these three foliations, the following polynomial (called CAMACHO- SAD *polynomial of the homogeneous foliation* $\mathcal{H}$)

$$\mathrm{CS}_{\mathcal{H}}(\lambda) = \prod_{s \in \mathrm{Sing}\mathcal{H} \cap L_\infty} (\lambda - \mathrm{CS}(\mathcal{H}, L_\infty, s)).$$

The following table summarizes the types and the CAMACHO- SAD polynomials of the foliations $\mathcal{H}_1^2$, $\mathcal{H}_2^2(0,0)$ and $\mathcal{H}_3^2(-2)$.

5 Pre-foliations of co-degree 1 whose associated foliation is reduced convex

We now give the proofs of Theorem E and Propositions F and G stated in the Introduction.

Proof of Theorem E Since by hypothesis $\mathcal{F}$ is reduced convex, all its singularities are non-degenerate ([4, Lemma 6.8]). According to [2, Lemma 2.2], the discriminant of $\mathrm{Leg}\mathcal{F}$ then consists of the lines dual to the radial singularities of $\mathcal{F}$. The first assertion of Lemma 2.1 therefore implies that

$$\Delta(\mathrm{Leg}\mathscr{F}) = \check{\Sigma}_{\mathcal{F}}^{\mathrm{rad}} \cup \check{\Sigma}_{\mathcal{F}}^{\ell}.$$

To show that the curvature of $\mathrm{Leg}\mathscr{F}$ is identically zero, it suffices therefore to show that it is holomorphic along the dual line of every point of $\Sigma_{\mathcal{F}}^{\mathrm{rad}} \cup \Sigma_{\mathcal{F}}^{\ell}$. Let s be an arbitrary point of $\Sigma_{\mathcal{F}}^{\mathrm{rad}} \cup \Sigma_{\mathcal{F}}^{\ell}$. Denote by $\nu = \tau(\mathcal{F}, s)$ the tangency order of $\mathcal{F}$ with a generic line passing through s; then $\nu - 1$ denotes the radiality order of s, and $s \in \Sigma_{\mathcal{F}}^{\mathrm{rad}}$ if and only if $\nu \geq 2$, *see* [4, §1.3]. By [13, Proposition 3.3], locally near the line $\check{s}$ dual to s, we can decompose $\mathrm{Leg}\mathcal{F}$ as $\mathrm{Leg}\mathcal{F} = \mathcal{W}_\nu \boxtimes \mathcal{W}_{d-\nu-1}$, where $\mathcal{W}_\nu$ is an irreducible ν-web leaving $\check{s}$ invariant and whose discriminant $\Delta(\mathcal{W}_\nu)$ has minimal multiplicity $\nu - 1$ along $\check{s}$, and where $\mathcal{W}_{d-\nu-1}$ is a

$(d-\nu-1)$-web transverse to $\check{s}$. Furthermore, the convexity of $\mathcal{F}$ implies, by an argument of the proof of [13, Theorem 4.2], that the web $\mathcal{W}_{d-\nu-1}$ is regular near $\check{s}$, *i.e.* that through a generic point of $\check{s}$ pass $(d-\nu-1)$ distinct tangent lines to $\mathcal{W}_{d-\nu-1}$.

Thus, near the line $\check{s}$, we have the decomposition

$$\mathrm{Leg}\mathscr{F} = \mathrm{Leg}\ell \boxtimes \mathcal{W}_\nu \boxtimes \mathcal{W}_{d-\nu-1}. \tag{5.1}$$

We now distinguish two cases:

1. If $s \in \ell$ then $\check{s}$ is invariant by $\mathrm{Leg}\ell$; by applying Theorem 1 of [13] if $\nu = 1$ and Proposition 3.9 if $\nu \geq 2$, it follows that $K(\mathrm{Leg}\mathscr{F})$ is holomorphic along $\check{s}$.

2. Assume that $s \notin \ell$; then $s \in \Sigma_{\mathcal{F}}^{\mathrm{rad}} \setminus \Sigma_{\mathcal{F}}^{\ell}$. In this case the radial foliation $\mathrm{Leg}\ell$ is transverse to $\check{s}$. From the above discussion, the $(d-\nu)$-web $\mathcal{W}_{d-\nu} := \mathrm{Leg}\ell \boxtimes \mathcal{W}_{d-\nu-1}$ is therefore also transverse to $\check{s}$ and we have $\mathrm{Leg}\mathscr{F} = \mathcal{W}_\nu \boxtimes \mathcal{W}_{d-\nu}$. Moreover, since ℓ is $\mathcal{F}$-invariant, $\mathrm{Tang}(\mathrm{Leg}\ell, \mathrm{Leg}\mathcal{F}) = \check{\Sigma}_{\mathcal{F}}^{\ell}$ (*cf.* proof of Lemma 2.1); in particular, $\mathrm{Tang}(\mathrm{Leg}\ell, \mathcal{W}_{d-\nu-1}) \subset \check{\Sigma}_{\mathcal{F}}^{\ell}$ and therefore $\check{s} \not\subset \mathrm{Tang}(\mathrm{Leg}\ell, \mathcal{W}_{d-\nu-1})$. It follows that the web $\mathcal{W}_{d-\nu}$ is regular near $\check{s}$, because $\mathcal{W}_{d-\nu-1}$ is so. As a consequence the curvature of $\mathrm{Leg}\mathscr{F}$ is holomorphic along $\check{s}$ by applying [13, Proposition 2.6]. □

The following proposition plays an important role in the proofs of Propositions F and G.

Proposition 5.1 *Let $\mathcal{F}$ be a reduced convex foliation of degree $d-1$ on $\mathbb{P}^2_{\mathbb{C}}$ with $d \geq 3$. Let ℓ be a line of $\mathbb{P}^2_{\mathbb{C}}$ which is not invariant by $\mathcal{F}$. Assume that $\mathcal{G}_{\mathcal{F}}(\ell)$ is equal to the dual line of a singularity s of $\mathcal{F}$ (necessarily $s \notin \ell$) such that $\tau(\mathcal{F}, s) = d-2$. Then the d-web $\mathrm{Leg}(\ell \boxtimes \mathcal{F})$ is flat.*

Remark 5.2 For $d = 3$ (resp. $d > 3$), the equality $\tau(\mathcal{F}, s) = d-2$ means that the singularity s of $\mathcal{F}$ is non-radial (resp. radial of order $d-3$).

The proof of Proposition 5.1 is based on the following two results.

Theorem 5.3 *Let $\mathscr{F} = \ell \boxtimes \mathcal{F}$ be a pre-foliation of co-degree 1 and degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$. Assume that the foliation $\mathcal{F}$ is reduced convex and that the line ℓ is not invariant by $\mathcal{F}$. Then, the curvature of $\mathrm{Leg}\mathscr{F}$ is holomorphic on $\check{\mathbb{P}}^2_{\mathbb{C}} \setminus \mathcal{G}_{\mathcal{F}}(\ell)$. In particular, the d-web $\mathrm{Leg}\mathscr{F}$ is flat if and only if $K(\mathrm{Leg}\mathscr{F})$ is holomorphic along $\mathcal{G}_{\mathcal{F}}(\ell)$.*

Proof It suffices to argue as in the proof of Theorem E. Indeed, first, the equality $\Delta(\mathrm{Leg}\mathcal{F}) = \check{\Sigma}_{\mathcal{F}}^{\mathrm{rad}}$ and the second assertion of Lemma 2.1 ensure that

$$\Delta(\mathrm{Leg}\mathscr{F}) = \check{\Sigma}_{\mathcal{F}}^{\mathrm{rad}} \cup \check{\Sigma}_{\mathcal{F}}^{\ell} \cup \mathcal{G}_{\mathcal{F}}(\ell).$$

Then, let $s \in \Sigma_{\mathcal{F}}^{\mathrm{rad}} \cup \Sigma_{\mathcal{F}}^{\ell}$ be such that $\check{s} \not\subset \mathcal{G}_{\mathcal{F}}(\ell)$ and let $\nu := \tau(\mathcal{F}, s)$; near the line $\check{s}$, the d-web $\mathrm{Leg}\mathscr{F}$ can be decomposed into the form (5.1). If $s \in \ell$, we can argue as in the case ***1.*** to deduce that $K(\mathrm{Leg}\mathscr{F})$ is holomorphic along $\check{s}$. If $s \notin \ell$, then $\check{s} \not\subset \mathcal{G}_{\mathcal{F}}(\ell) \cup \check{\Sigma}_{\mathcal{F}}^{\ell} = \mathrm{Tang}(\mathrm{Leg}\ell, \mathrm{Leg}\mathcal{F})$ and we can argue as in the case ***2.*** to deduce the same conclusion. It follows that $K(\mathrm{Leg}\mathscr{F})$ is holomorphic on $\left(\check{\Sigma}_{\mathcal{F}}^{\mathrm{rad}} \cup \check{\Sigma}_{\mathcal{F}}^{\ell}\right) \setminus \mathcal{G}_{\mathcal{F}}(\ell) = \Delta(\mathrm{Leg}\mathscr{F}) \setminus \mathcal{G}_{\mathcal{F}}(\ell)$ and therefore on $\check{\mathbb{P}}^2_{\mathbb{C}} \setminus \mathcal{G}_{\mathcal{F}}(\ell)$. □

Proposition 5.4 *Let $\mathcal{W}_{d-2}$ be a germ of $(d-2)$-web on $(\mathbb{C}^2, 0)$. Assume that $\Delta(\mathcal{W}_{d-2})$ has an irreducible component C totally invariant by $\mathcal{W}_{d-2}$. Let $\mathcal{W}_2$ be a germ of a 2-web on $(\mathbb{C}^2, 0)$ transverse to C. Then, the curvature of the d-web $\mathcal{W} = \mathcal{W}_{d-2} \boxtimes \mathcal{W}_2$ is holomorphic along C if and only if the curvature of the $(d-2)$-web $\mathcal{W}_{d-2}$ is holomorphic along C.*

Proof Locally, in a neighborhood of a generic point of C, we can write $\mathcal{W}_{d-2} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_{d-2}$ and $\mathcal{W}_2 = \mathcal{F}_1^{'} \boxtimes \mathcal{F}_2^{'}$, where each $\mathcal{F}_i$ is a foliation leaving C invariant, and $\mathcal{F}_1^{'}$ and $\mathcal{F}_2^{'}$ are foliations transverse to C. We have

$$K(\mathcal{W}) = K(\mathcal{W}_{d-2}) + \sum_{i=1}^{d-2} K(\mathcal{F}_i \boxtimes \mathcal{W}_2) + \sum_{k=1}^{2} \sum_{1\leq i<j\leq d-2} K(\mathcal{F}_i \boxtimes \mathcal{F}_j \boxtimes \mathcal{F}_k^{'}).$$

Now, $K(\mathcal{F}_i \boxtimes \mathcal{W}_2)$ and $K(\mathcal{F}_i \boxtimes \mathcal{F}_j \boxtimes \mathcal{F}_k^{'})$ are holomorphic along C by applying [13, Theorem 1]. Hence, $K(\mathcal{W})$ is holomorphic on C if and only if $K(\mathcal{W}_{d-2})$ is so. □

Proof of Proposition ***5.1*** According to Theorem 5.3, it suffices to prove that the curvature of $\mathrm{Leg}(\ell \boxtimes \mathcal{F})$ is holomorphic on $\mathcal{G}_{\mathcal{F}}(\ell) = \check{s}$. By [13, Proposition 3.3] and the equality $\tau(\mathcal{F}, s) = d-2$, near the line $\check{s}$, we can decompose the web $\mathrm{Leg}\mathcal{F}$ as $\mathrm{Leg}\mathcal{F} = \mathcal{W}_{d-2} \boxtimes \mathcal{W}_1$, where $\mathcal{W}_{d-2}$ is a $(d-2)$-web having $\check{s}$ as a totally invariant curve with $\mathrm{mult}(\Delta(\mathcal{W}_{d-2}), \check{s}) = d-3$, and $\mathcal{W}_1$ is a foliation transverse to $\check{s}$. Thus, $\mathrm{Leg}(\ell \boxtimes \mathcal{F}) = \mathcal{W}_{d-2} \boxtimes \mathcal{W}_2$, where $\mathcal{W}_2 := \mathrm{Leg}\ell \boxtimes \mathcal{W}_1$ is transverse to $\check{s}$, because $s \notin \ell$. By [13, Proposition 2.6], $K(\mathcal{W}_{d-2})$ is holomorphic on $\check{s}$, and by Proposition 5.4, the same is true for $K(\mathrm{Leg}(\ell \boxtimes \mathcal{F}))$. □

Proof of Proposition F The FERMAT foliation $\mathcal{F}_0^{d-1}$ is given in homogeneous coordinates by the 1-form

$$\overline{\Omega}_0^{d-1} = x^{d-1}(y\mathrm{d}z - z\mathrm{d}y) + y^{d-1}(z\mathrm{d}x - x\mathrm{d}z) + z^{d-1}(x\mathrm{d}y - y\mathrm{d}x).$$

It has the following $3(d-1)$ invariant lines:

$$x=0,\ y=0,\ z=0,\ y=\zeta^k x,\ y=\zeta^k z,\ x=\zeta^k z,$$

where

$$k \in \{0,\ldots,d-3\} \text{ and } \zeta = \exp(\tfrac{2i\pi}{d-2}).$$

Since the coordinates x, y and z play a symmetric role and since ℓ is not invariant by $\mathcal{F}_0^{d-1}$, we can assume that $\ell = \{\alpha x + \beta y - z = 0\}$ with $\beta \neq 0$. Then $\overline{\mathcal{O}(\ell \boxtimes \mathcal{F}_0^{d-1})}$ contains the following homogeneous pre-foliations:

$$\mathscr{H}_1 = \{y - \alpha x = 0\} \boxtimes \mathcal{H}_1^{d-1},\ \mathscr{H}_2 = \{y - \beta x = 0\} \boxtimes \mathcal{H}_1^{d-1},$$
$$\mathscr{H}_3 = \big\{x - (\alpha+\beta)y = 0\big\} \boxtimes \mathcal{H}_0^{d-1}.$$

Indeed, $\ell \boxtimes \mathcal{F}_0^{d-1}$ is described in the affine chart $z=1$ by $\omega = (\alpha x + \beta y - 1)\overline{\omega}_0^{d-1}$; putting $\varphi_1 = \left(\frac{x}{y}, \frac{\varepsilon}{y}\right)$, $\varphi_2 = \left(\frac{\varepsilon}{y}, \frac{x}{y}\right)$ and $\varphi_3 = \left(\frac{y+\varepsilon}{x}, \frac{y}{x}\right)$, we obtain that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} y^{d+2} \varphi_1^* \omega = (y - \alpha x)\omega_1^{d-1},\ \lim_{\varepsilon\to 0} \varepsilon^{-1} y^{d+2} \varphi_2^* \omega = (\beta x - y)\omega_1^{d-1},$$
$$\lim_{\varepsilon\to 0} \varepsilon^{-1} x^{d+2} \varphi_3^* \omega = \Big((\alpha+\beta)y - x\Big)\omega_0^{d-1}.$$

The hypothesis that $\mathrm{Leg}(\ell \boxtimes \mathcal{F}_0^{d-1})$ is flat implies that the webs $\mathrm{Leg}\mathscr{H}_i$ $(i=1,2,3)$ are also flat. Let us show that the flatness of $\mathrm{Leg}\mathscr{H}_1$ and $\mathrm{Leg}\mathscr{H}_2$ implies that, up to linear conjugation,

$$(\alpha, \beta) \in E := \Big\{(0,\xi), (1,1), (1,\xi), (\xi,\xi)\Big\}, \quad \text{where } \xi = \exp\left(\tfrac{i\pi}{d-2}\right).$$

First of all, the d-web $\mathrm{Leg}\mathscr{H}_1$, resp. $\mathrm{Leg}\mathscr{H}_2$, is flat if and only if (*cf.* proof of Lemma 4.1)

$$\alpha(\alpha^{d-2}-1)(\alpha^{d-2}+1) = 0, \qquad \text{resp. } (\beta^{d-2}-1)(\beta^{d-2}+1) = 0,$$

i.e. if and only if $\alpha \in \{0, \zeta^k, \xi\zeta^k,\ k = 0, \ldots, d-3\}$, resp. $\beta \in \{\zeta^k, \xi\zeta^k,\ k = 0, \ldots, d-3\}$. If $\alpha = 0$ then $\beta \neq \zeta^k$, because otherwise ℓ would be invariant by $\mathcal{F}_0^{d-1}$. It follows that

$$(\alpha, \beta) \in \Big\{(0, \xi\zeta^k), (\zeta^k, \zeta^{k'}), (\zeta^k, \xi\zeta^{k'}), (\xi\zeta^k, \zeta^{k'}), (\xi\zeta^k, \xi\zeta^{k'}),\ \ k, k' = 0, \ldots, d-3\Big\}.$$

If, for $k, k' \in \{0, \ldots, d-3\}$,

$$(\alpha, \beta) = (0, \xi\zeta^k), \qquad \text{resp. } (\alpha, \beta) \in \Big\{(\zeta^k, \zeta^{k'}), (\zeta^k, \xi\zeta^{k'}), (\xi\zeta^k, \xi\zeta^{k'})\Big\},$$
$$\text{resp.} (\alpha, \beta) = (\xi\zeta^k, \zeta^{k'}),$$

then by conjugating ω by $\left(x, \frac{y}{\zeta^k}\right)$, resp. $\left(\frac{x}{\zeta^k}, \frac{y}{\zeta^{k'}}\right)$, resp. $\left(\frac{y}{\zeta^k}, \frac{x}{\zeta^{k'}}\right)$, we reduce ourselves to $(\alpha, \beta) = (0, \xi)$, resp. $(\alpha, \beta) \in \{(1, 1), (1, \xi), (\xi, \xi)\}$, resp. $(\alpha, \beta) = (1, \xi)$. Thus, up to conjugation, (α, β) belongs to E.

Moreover, according to Example 3.23, the flatness of $\mathrm{Leg}\mathscr{H}_3$ is equivalent to

$$0 = (\alpha + \beta)\Big((\alpha + \beta)^{d-2} - 1\Big)\Big((\alpha + \beta)^{d-2} - 2(d-2)\Big) =: f_d(\alpha, \beta).$$

Since

$$f_d(0, \xi) = 2\xi(2d-3) \neq 0,\ f_d(1, 1) = 4(2^{d-2} - 1)(2^{d-3} - d + 2) = 0 \Longleftrightarrow d \in \{3, 4\},$$
$$f_d(\xi, \xi) = 2\xi(2^{d-2} + 1)(2^{d-2} + 2d - 4) \neq 0,$$
$$f_d(1, \xi) = (\xi + 1)\Big((\xi + 1)^{d-2} - 1\Big)\Big((\xi + 1)^{d-2} - 2(d-2)\Big) = 0 \Longleftrightarrow d = 3,$$

we deduce that $d \in \{3, 4\}$ and, up to conjugation,

$$(\alpha, \beta) \in \{(1, 1), (1, -1)\} \text{ if } d = 3 \qquad \text{and} \qquad (\alpha, \beta) = (1, 1) \text{ if } d = 4,$$

i.e., putting $\ell_1 = \{x + y - z = 0\}$ and $\ell_2 = \{x - y - z = 0\}$, we have $\ell \in \{\ell_1, \ell_2\}$ if $d = 3$ and $\ell = \ell_1$ if $d = 4$.

Even in the case $d = 3$, we can take $\ell = \ell_1$, because $\ell_1 \boxtimes \mathcal{F}_0^{d-1}$ and $\ell_2 \boxtimes \mathcal{F}_0^{d-1}$ are conjugate, via $\phi = [z : y : x]$. Note that $\ell_1 = (s_1 s_2 s_3)$, where $s_1 = [1 : 0 : 1]$, $s_2 = [0 : 1 : 1]$ and $s_3 = [-1 : 1 : 0]$: the points s_1 and s_2 are singular for $\mathcal{F}_0^{d-1}$, and $s_3 \in \mathrm{Sing}\mathcal{F}_0^{d-1}$ if and only if d is even; in particular the point s_3 is singular for $\mathcal{F}_0^3$ but not for $\mathcal{F}_0^2$.

Finally, a straightforward computation shows that $\mathcal{G}_{\mathcal{F}_0^2}(\ell_1) = \check{s}_4$ and $\mathcal{G}_{\mathcal{F}_0^3}(\ell_1) = \check{s}_5$, where $s_4 = [1 : 1 : 0]$ is a non-radial singularity of $\mathcal{F}_0^2$ and $s_5 = [-1 : -1 : 1]$ is a radial singularity of order 1 of $\mathcal{F}_0^3$. We can then apply Proposition 5.1 to conclude that the webs $\mathrm{Leg}(\ell_1 \boxtimes \mathcal{F}_0^2)$ and $\mathrm{Leg}(\ell_1 \boxtimes \mathcal{F}_0^3)$ are flat. □

Proof of Proposition G The HESSE foliation $\mathcal{F}_H^4$ is described in homogeneous coordinates by the 1-form

$$\Omega_H^4 = yz(2x^3 - y^3 - z^3)\mathrm{d}x + xz(2y^3 - x^3 - z^3)\mathrm{d}y + xy(2z^3 - x^3 - y^3)\mathrm{d}z.$$

Its 12 invariant lines are given by

$$xyz(x + y + z)(\zeta x + y + z)(x + \zeta y + z)(x + y + \zeta z)(\zeta^2 x + y + z)$$
$$(x + \zeta^2 y + z)(x + y + \zeta^2 z)(\zeta^2 x + \zeta y + z)(\zeta x + \zeta^2 y + z) = 0,$$

where $\zeta = \exp(\frac{2i\pi}{3})$. As above, we can assume that $\ell = \{\alpha x + \beta y - z = 0\}$ with $\beta \neq 0$. Then the closure of the $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$-orbit of $\ell \boxtimes \mathcal{F}_H^4$ contains the following three homogeneous pre-foliations:

$$\mathscr{H}_1 = \{y - \alpha x = 0\} \boxtimes \mathcal{H}_4^4, \quad \mathscr{H}_2 = \{y - \beta x = 0\} \boxtimes \mathcal{H}_4^4, \quad \mathscr{H}_3 = \big\{a\,x + by = 0\big\} \boxtimes \mathcal{H}_4^4,$$

where $a = \alpha + \beta - 1$ and $b = \alpha + \zeta^2\beta - \zeta$. Indeed, in the affine chart $z = 1$, the pre-foliation $\ell \boxtimes \mathcal{F}_H^4$ is given by $\omega = (\alpha x + \beta y - 1)\omega_H^4$; putting $\psi_1 = \left(\frac{x}{y}, \frac{\varepsilon}{y}\right)$, $\psi_2 = \left(\frac{\varepsilon}{y}, \frac{x}{y}\right)$ and $\psi_3 = \left(\frac{x+y}{x+\zeta y+\varepsilon}, \frac{x+\zeta^2 y}{x+\zeta y+\varepsilon}\right)$, a straightforward computation shows that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} y^7 \psi_1^*\omega = (\alpha x - y)\omega_4^4, \; \lim_{\varepsilon\to 0} \varepsilon^{-1} y^7 \psi_2^*\omega = (\beta x - y)\omega_4^4,$$
$$\lim_{\varepsilon\to 0} \varepsilon^{-1} (x + \zeta y + \varepsilon)^7 \psi_3^*\omega = -9\zeta(a\,x + by)\omega_4^4.$$

Since the 5-web $\mathrm{Leg}(\ell \boxtimes \mathcal{F}_H^4)$ is flat by hypothesis, so are the 5-webs $\mathrm{Leg}\mathscr{H}_i$, $i = 1, 2, 3$. Now, according to Example 3.24, for any line ℓ_0 passing through the origin, $\mathrm{Leg}(\ell_0 \boxtimes \mathcal{H}_4^4)$ is flat if and only if $\ell_0 = \{x = 0\}$ or $\ell_0 = \{y - \rho x = 0\}$ with $\rho(\rho^3 - 1)(\rho^3 + 1) = 0$, *i.e.* $\rho \in E := \{0, \zeta^k, -\zeta^k,\ k = 0, 1, 2\}$. Therefore, the flatness of $\mathrm{Leg}\mathscr{H}_1$ (resp. $\mathrm{Leg}\mathscr{H}_2$) is equivalent to $\alpha \in E$ (resp. $\beta \in E \setminus \{0\}$ because $\beta \neq 0$). Note that $(\alpha, \beta) \neq (-\zeta^k, -\zeta^{k'})$, for otherwise ℓ would be invariant by $\mathcal{F}_H^4$. As a result

$$(\alpha, \beta) \in \Big\{(0, \zeta^k), (0, -\zeta^k), (\zeta^k, \zeta^{k'}), (\zeta^k, -\zeta^{k'}), (-\zeta^k, \zeta^{k'}),\ \ k, k' = 0, 1, 2\Big\}.$$

If, for $k, k' \in \{0, 1, 2\}$,

$$(\alpha, \beta) \in \Big\{(0, \zeta^k), (0, -\zeta^k)\Big\}, \text{resp.} (\alpha, \beta) \in \Big\{(\zeta^k, \zeta^{k'}), (\zeta^k, -\zeta^{k'})\Big\},$$
$$\text{resp.} (\alpha, \beta) = (-\zeta^k, \zeta^{k'}),$$

then by conjugating ω by $\left(x, \frac{y}{\zeta^k}\right)$, resp. $\left(\frac{x}{\zeta^k}, \frac{y}{\zeta^{k'}}\right)$, resp. $\left(\frac{y}{\zeta^k}, \frac{x}{\zeta^{k'}}\right)$, we reduce ourselves to $(\alpha, \beta) \in \{(0, 1), (0, -1)\}$, resp. $(\alpha, \beta) \in \{(1, 1), (1, -1)\}$, resp. $(\alpha, \beta) = (1, -1)$. It follows that, up to linear conjugation,

$$(\alpha, \beta) \in F := \Big\{(0, 1), (0, -1), (1, 1), (1, -1)\Big\}.$$

Now, for $(\alpha, \beta) \in F$, $\mathrm{Leg}\mathscr{H}_3$ is flat if and only if $(\alpha, \beta) = (0, 1)$, since putting $\tau(\alpha, \beta) = -\frac{a}{b}$, we have

$$\tau(0, 1) = 0 \in E, \quad \tau(0, -1) = 2 \notin E, \quad \tau(1, 1) = \tfrac{\zeta^2}{2} \notin E, \quad \tau(1, -1) = \tfrac{1}{2} \notin E.$$

Therefore, up to conjugation, $(\alpha, \beta) = (0, 1)$, *i.e.* $\ell = \ell_0 := \{y - z = 0\}$; this line passes through four singular points of $\mathcal{F}_H^4$, namely the points $s_1 = [1 : 0 : 0]$, $s_2 = [1 : 1 : 1]$, $s_3 = [\zeta : 1 : 1]$ and $s_4 = [\zeta^2 : 1 : 1]$.

Finally, it is easy to check that $\mathcal{G}_{\mathcal{F}_H^4}(\ell_0)$ is equal to the dual line of the point $s_5 = [0 : -1 : 1]$ which is a radial singularity of order 2 of $\mathcal{F}_H^4$. This implies, by Proposition 5.1, that the 5-web $\mathrm{Leg}(\ell_0 \boxtimes \mathcal{F}_H^4)$ is flat. □

6 Pre-foliations of type $(1,3)$ whose associated foliation has only non-degenerate singularities

In this section, we prove Theorem H stated in the Introduction. To do this, we need two preliminary results, the first of which holds in any degree.

Let us first recall that in §5 we have proved Propositions F and G by reducing to the homogeneous case; in fact this argument is implicitly based on the following proposition.

Proposition 6.1 *Let $\mathscr{F} = \ell \boxtimes \mathcal{F}$ be a pre-foliation of co-degree 1 and degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$. Assume that the foliation $\mathcal{F}$ has an invariant line D and that all its singularities on D are non-degenerate. There is a homogeneous pre-foliation $\mathscr{H} = \ell_0 \boxtimes \mathcal{H}$ of co-degree 1 and degree d on $\mathbb{P}^2_{\mathbb{C}}$ such that:*

(i) $\mathscr{H} \in \overline{\mathcal{O}(\mathscr{F})}$ and $\mathcal{H} \in \overline{\mathcal{O}(\mathcal{F})}$;
(ii) *if $\ell = D$ (resp. $\ell \neq D$), then $\ell_0 = L_\infty$ (resp. $\ell_0 \neq L_\infty$);*
(iii) *D is invariant by $\mathcal{H}$;*
(iv) $\mathrm{Sing}\mathcal{H} \cap D = \mathrm{Sing}\mathcal{F} \cap D$;
(v) $\forall\, s \in \mathrm{Sing}\mathcal{H} \cap D,\ \mu(\mathcal{H}, s) = 1$ *and* $\mathrm{CS}(\mathcal{H}, D, s) = \mathrm{CS}(\mathcal{F}, D, s)$.

If, moreover, $\mathrm{Leg}\mathscr{F}$ *(resp.* $\mathrm{Leg}\mathcal{F}$*) is flat, then* $\mathrm{Leg}\mathscr{H}$ *(resp.* $\mathrm{Leg}\mathcal{H}$*) is also flat.*

This proposition is an analogue for co-degree one pre-foliations of Proposition 6.4 of [4] on foliations of $\mathbb{P}^2_{\mathbb{C}}$.

Proof Choose a homogeneous coordinate system $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ such that $D = L_\infty = (z = 0)$. Since D is $\mathcal{F}$-invariant, $\mathcal{F}$ is defined in the affine chart $z = 1$ by a 1-form ω of type

$$\omega = \sum_{i=0}^{d-1} (A_i(x, y)\mathrm{d}x + B_i(x, y)\mathrm{d}y),$$

where $A_i,\ B_i$ are homogeneous polynomials of degree i. According to [4, Proposition 6.4], since all the singularities of $\mathcal{F}$ on D are non-degenerate, the 1-form $\omega_{d-1} = A_{d-1}(x, y)\mathrm{d}x + B_{d-1}(x, y)\mathrm{d}y$ defines a homogeneous foliation $\mathcal{H}$ of degree $d-1$ on $\mathbb{P}^2_{\mathbb{C}}$ belonging to $\overline{\mathcal{O}(\mathcal{F})}$ and satisfying the stated properties (iii), (iv) and (v).

Now, write $\ell = \{\alpha x + \beta y + \gamma z = 0\}$; in homogeneous coordinates, $\mathscr{F}$, resp. $\mathcal{H}$, is given by

$$\Omega_{d+1} = (\alpha x + \beta y + \gamma z) \sum_{i=0}^{d-1} z^{d-i-1} \Big(A_i(x, y)(z\mathrm{d}x - x\mathrm{d}z) + B_i(x, y)(z\mathrm{d}y - y\mathrm{d}z) \Big),$$

$$\text{resp.}\Omega_d = A_{d-1}(x, y)(z\mathrm{d}x - x\mathrm{d}z) + B_{d-1}(x, y)(z\mathrm{d}y - y\mathrm{d}z).$$

Putting $\varphi = \varphi_\varepsilon = \left[\frac{x}{\varepsilon} : \frac{y}{\varepsilon} : z\right]$, we see that if $(\alpha, \beta) = (0, 0)$, resp. $(\alpha, \beta) \neq (0, 0)$, then

$$\lim_{\varepsilon \to 0} \varepsilon^d \gamma^{-1} \varphi^* \Omega_{d+1} = z\Omega_d, \qquad \text{resp. } \lim_{\varepsilon \to 0} \varepsilon^{d+1} \varphi^* \Omega_{d+1} = (\alpha x + \beta y)\Omega_d.$$

It follows that the closure of the $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$-orbit of $\mathscr{F}$ contains the homogeneous pre-foliation $\mathscr{H} = \ell_0 \boxtimes \mathcal{H}$, where $\ell_0 = L_\infty$ if $\ell = D$ and $\ell_0 = \{\alpha x + \beta y = 0\} \neq L_\infty$ if $\ell \neq D$. □

The following technical lemma is an analogue for pre-foliations of type $(1, 3)$ of Lemma 6.7 of [4] on foliations of degree 3. It plays a key role in the proof of Theorem H.

Lemma 6.2 *Let $\mathscr{F} = \ell \boxtimes \mathcal{F}$ be a pre-foliation of co-degree* 1 *and degree* 3 *on* $\mathbb{P}^2_{\mathbb{C}}$*. Assume that the* 3*-web* $\mathrm{Leg}\mathscr{F}$ *is flat and that the foliation* $\mathcal{F}$ *has a non-degenerate singularity* m *satisfying* $\mathrm{BB}(\mathcal{F}, m) \neq 4$*. Then, through the point* m *pass exactly two* $\mathcal{F}$*-invariant lines.*

Proof The hypotheses $\mu(\mathcal{F}, m) = 1$ and $\mathrm{BB}(\mathcal{F}, m) \neq 4$ ensure the existence of an affine chart (x, y) of $\mathbb{P}^2_{\mathbb{C}}$ in which $m = (0, 0)$ and $\mathcal{F}$ is defined by a 1-form ω_0 of type $\omega_0 = \omega_{0,1} + \omega_{0,2} + \omega_{0,3}$, where

$$\omega_{0,1} = \lambda y\mathrm{d}x + \mu x\mathrm{d}y, \qquad \omega_{0,2} = \left(\sum_{i=0}^{2} a_i x^{2-i} y^i\right)\mathrm{d}x + \left(\sum_{i=0}^{2} b_i x^{2-i} y^i\right)\mathrm{d}y,$$

$$\omega_{0,3} = \left(\sum_{i=0}^{2} c_i x^{2-i} y^i\right)(x\mathrm{d}y - y\mathrm{d}x),$$

with $\lambda\mu(\lambda + \mu) \neq 0$.

The only lines passing through m and which can be invariant by $\mathcal{F}$ are $(x = 0)$ and $(y = 0)$. Indeed, denote by $\mathrm{R} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ the radial vector field centered at m; if $L = (ux + vy = 0)$ is $\mathcal{F}$-invariant, then $ux + vy$ must divide the tangent cone $\mathrm{C}_{\omega_{0,1}} := \omega_{0,1}(\mathrm{R}) = (\lambda + \mu)xy$, so that $u = 0$ or $v = 0$.

We will show that indeed $(x = 0)$ and $(y = 0)$ are invariant by $\mathcal{F}$, which will establish the lemma. We have to prove that $a_0 = b_2 = 0$, since the invariance by $\mathcal{F}$ of $(x = 0)$, resp. $(y = 0)$, is equivalent to the vanishing of b_2, resp. a_0.

If $\ell = \{\alpha x + \beta y + \gamma = 0\}$ then, in the affine chart (p, q) of $\check{\mathbb{P}}^2_{\mathbb{C}}$ corresponding to the line $\{y = px - q\} \subset \mathbb{P}^2_{\mathbb{C}}$, the 3-web $\mathrm{Leg}\mathscr{F}$ is described by the symmetric 3-form

$$\check{\omega} = \big((\gamma - \beta q)\mathrm{d}p + (\alpha + \beta p)\mathrm{d}q\big)\check{\omega}_0,$$

where

$$\check{\omega}_0 = \mu\, p\mathrm{d}p\mathrm{d}q + (a_0 + b_0 p + c_0 q)\mathrm{d}q^2 + \Big(\lambda\,\mathrm{d}p + (a_1 + b_1 p + c_1 q)\mathrm{d}q\Big)(p\mathrm{d}q - q\mathrm{d}p)$$
$$+ \big(a_2 + b_2 p + c_2 q\big)\big(p\mathrm{d}q - q\mathrm{d}p\big)^2.$$

Assume by contradiction that $a_0 \neq 0$. Consider the family of automorphisms $\varphi = \varphi_\varepsilon = (a_0\varepsilon p, a_0\varepsilon^2 q)$. We see that if $\gamma \neq 0$, resp. $\gamma = 0$ and $\alpha \neq 0$, resp. $\gamma = \alpha = 0$, then

$$\lim_{\varepsilon\to 0} \varepsilon^{-5}\gamma^{-1}a_0^{-4}\varphi^*\check{\omega} = \theta_1\eta, \qquad \text{resp. } \lim_{\varepsilon\to 0} \varepsilon^{-6}\alpha^{-1}a_0^{-4}\varphi^*\check{\omega} = \theta_2\eta,$$
$$\text{resp. } \lim_{\varepsilon\to 0} \varepsilon^{-7}\beta^{-1}a_0^{-5}\varphi^*\check{\omega} = \theta_3\eta,$$

where

$$\theta_1 = \mathrm{d}p, \quad \theta_2 = \mathrm{d}q, \quad \theta_3 = p\mathrm{d}q - q\mathrm{d}p, \quad \eta = -\lambda\, q\mathrm{d}p^2 + (\lambda + \mu)p\mathrm{d}p\mathrm{d}q + \mathrm{d}q^2.$$

For $i = 1, 2, 3$, put $\mathcal{W}_3^{(i)} = \mathcal{F}_i \boxtimes \mathcal{W}_2$, where $\mathcal{W}_2$ (resp. $\mathcal{F}_i$) denotes the 2-web (resp. the foliation) defined by η (resp. by θ_i). It follows that if $\gamma \neq 0$, resp. $\gamma = 0$ and $\alpha \neq 0$, resp. $\gamma = \alpha = 0$, then the closure of the $\mathrm{Aut}(\check{\mathbb{P}}^2_{\mathbb{C}})$-orbit of $\mathrm{Leg}\mathscr{F}$ contains the 3-web $\mathcal{W}_3^{(1)}$, resp. $\mathcal{W}_3^{(2)}$, resp. $\mathcal{W}_3^{(3)}$. Now, since $\mathrm{Leg}\mathscr{F}$ is flat by hypothesis, every 3-web belonging to $\overline{\mathcal{O}(\mathrm{Leg}\mathscr{F})}$ is also flat. Therefore, to obtain a contradiction, it suffices to show that for every $i = 1, 2, 3$, $\mathcal{W}_3^{(i)}$ is not flat. Since $\Delta(\eta) = f(p, q) := 4\lambda q + (\lambda + \mu)^2 p^2$, it suffices again to show that for every $i = 1, 2, 3$, the curvature of $\mathcal{W}_3^{(i)}$ is not holomorphic along the component $\mathcal{C} = \{f(p, q) = 0\} \subset \Delta(\mathcal{W}_2)$, which is a parabola, because $\lambda(\lambda + \mu) \neq 0$.

First of all, let us note that $\mathcal{C}$ is not invariant by $\mathcal{W}_2$, since putting $\eta_0 = (\lambda+\mu)p\mathrm{d}p+2\mathrm{d}q$, we have

$$\eta\big|_{\mathcal{C}} = \left(\frac{\eta_0}{2}\right)^2 \qquad \text{and} \qquad \eta_0 \wedge \mathrm{d}f = -4\mu(\lambda+\mu)p\mathrm{d}p \wedge \mathrm{d}q \not\equiv 0.$$

Let us consider the case where $i \in \{1, 2\}$. Since

$$\eta_0 \wedge \theta_1\Big|_{\mathcal{C}} = -2\mathrm{d}p \wedge \mathrm{d}q \not\equiv 0 \quad \text{and} \quad \eta_0 \wedge \theta_2\Big|_{\mathcal{C}} = (\lambda+\mu)p\mathrm{d}p \wedge \mathrm{d}q \not\equiv 0,$$

we have $\mathcal{C} \not\subset \mathrm{Tang}(\mathcal{W}_2, \mathcal{F}_i)$. Therefore, according to [13, Theorem 1] (*cf.* [2, Theorem 1.1]), the curvature $K(\mathcal{W}_3^{(i)})$ is holomorphic on $\mathcal{C}$ if and only if $\mathcal{C}$ is invariant by $\mathcal{F}_i$, which is impossible, because each $\mathcal{F}_i$ is a pencil of lines and hence cannot admit a parabola as an invariant curve.

Let us now examine the case where $i = 3$. In this case $\mathcal{C} \subset \mathrm{Tang}(\mathcal{W}_2, \mathcal{F}_3)$ if and only if $\lambda = \mu$, because

$$\eta_0 \wedge \theta_3\Big|_{\mathcal{C}} = \frac{1}{2\lambda}(\lambda-\mu)(\lambda+\mu)p^2\mathrm{d}p \wedge \mathrm{d}q \equiv 0 \Longleftrightarrow \lambda = \mu.$$

If $\lambda \neq \mu$, then, as above, we can apply Theorem 1 of [13] and assert that $K(\mathcal{W}_3^{(3)})$ cannot be holomorphic on $\mathcal{C}$.

We therefore assume that $\lambda = \mu$ and prove that $K(\mathcal{W}_3^{(3)}) \not\equiv 0$. The pull-back of $\mathcal{W}_3^{(3)}$ by the rational map $\psi(p,q) = \big(p, \mu(q^2-p^2)\big)$ writes as $\psi^*\mathcal{W}_3^{(3)} = \mathcal{F}_3^{(1)} \boxtimes \mathcal{F}_3^{(2)} \boxtimes \mathcal{F}_3^{(3)}$, where

$$\mathcal{F}_3^{(1)} : (p^2+q^2)\mathrm{d}p - 2pq\mathrm{d}q = 0, \mathcal{F}_3^{(2)} : (p+q)\mathrm{d}p - 2q\mathrm{d}q = 0,$$
$$\mathcal{F}_3^{(3)} : (p-q)\mathrm{d}p - 2q\mathrm{d}q = 0.$$

Using formula (1.1), a direct computation leads to

$$\eta(\psi^*\mathcal{W}_3^{(3)}) = -\frac{p\mathrm{d}p}{q^2} + \frac{4\mathrm{d}q}{q} + \frac{\mathrm{d}(p^2-q^2)}{p^2-q^2},$$

so that

$$K(\psi^*\mathcal{W}_3^{(3)}) = \mathrm{d}\eta(\psi^*\mathcal{W}_3^{(3)}) = -\frac{2p}{q^3}\mathrm{d}p \wedge \mathrm{d}q \not\equiv 0,$$

hence $\psi^*K(\mathcal{W}_3^{(3)}) = K(\psi^*\mathcal{W}_3^{(3)}) \not\equiv 0$ and therefore $K(\mathcal{W}_3^{(3)}) \not\equiv 0$.

We have thus shown that $a_0 = 0$, which means that the line $(y = 0)$ is invariant by $\mathcal{F}$. Exchanging the roles of the coordinates x and y, the same argument shows that $b_2 = 0$, *i.e.* that the line $(x = 0)$ is also invariant by $\mathcal{F}$. □

Before starting the proof of Theorem H, let us recall (*cf.* [8]) that if $\mathcal{F}$ is a foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ then

$$\sum_{s \in \mathrm{Sing}\mathcal{F}} \mu(\mathcal{F}, s) = d^2+d+1 \qquad \text{and} \qquad \sum_{s \in \mathrm{Sing}\mathcal{F}} \mathrm{BB}(\mathcal{F}, s) = (d+2)^2. \qquad (6.1)$$

Proof of Theorem H Write $\mathrm{Sing}\mathcal{F} = \Sigma^1 \cup \Sigma^2$, where

$$\Sigma^1 = \{s \in \mathrm{Sing}\mathcal{F} : \mathrm{BB}(\mathcal{F}, s) = 4\} \qquad \text{and} \qquad \Sigma^2 = \mathrm{Sing}\mathcal{F} \setminus \Sigma^1.$$

For $i = 1, 2$, put $\kappa_i = \#\,\Sigma^i$. By hypothesis, we have $\deg \mathcal{F} = 2$ and, for any $s \in \mathrm{Sing}\mathcal{F}$, $\mu(\mathcal{F}, s) = 1$. Formulas (6.1) then give

$$\#\,\mathrm{Sing}\mathcal{F} = \kappa_1 + \kappa_2 = 7 \qquad \text{and} \qquad 4\kappa_1 + \sum_{s\in\Sigma^2} \mathrm{BB}(\mathcal{F}, s) = 16. \tag{6.2}$$

It follows that Σ^2 is non-empty. Let m be a point of Σ^2; by Lemma 6.2 through the point m pass exactly two $\mathcal{F}$-invariant lines $D_m^{(1)}$ and $D_m^{(2)}$. Then, for $i = 1, 2$, Proposition 6.1 ensures the existence of a homogeneous pre-foliation $\mathscr{H}_m^{(i)} = \ell_m^{(i)} \boxtimes \mathcal{H}_m^{(i)}$ of type $(1, 3)$ on $\mathbb{P}^2_{\mathbb{C}}$ belonging to $\overline{\mathcal{O}(\mathscr{F})}$ and such that the line $D_m^{(i)}$ is $\mathcal{H}_m^{(i)}$-invariant. Since $\mathrm{Leg}\mathscr{F}$ is flat by hypothesis, so are $\mathrm{Leg}\mathscr{H}_m^{(1)}$ and $\mathrm{Leg}\mathscr{H}_m^{(2)}$. Therefore, $\mathscr{H}_m^{(i)}$ $(i = 1, 2)$ is linearly conjugate to one of the eight models of Corollary 4.5. For $i = 1, 2$, Proposition 6.1 also ensures that

(𝔞) if $\ell \neq D_m^{(i)}$, then $\ell_m^{(i)} \neq L_\infty$;
(𝔟) $\mathrm{Sing}\mathcal{F} \cap D_m^{(i)} = \mathrm{Sing}\mathcal{H}_m^{(i)} \cap D_m^{(i)}$;
(𝔠) $\forall s \in \mathrm{Sing}\mathcal{H}_m^{(i)} \cap D_m^{(i)}$, $\mu(\mathcal{H}_m^{(i)}, s) = 1$ and $\mathrm{CS}(\mathcal{H}_m^{(i)}, D_m^{(i)}, s) = \mathrm{CS}(\mathcal{F}, D_m^{(i)}, s)$.

Since $\mathrm{CS}(\mathcal{F}, D_m^{(1)}, m)\mathrm{CS}(\mathcal{F}, D_m^{(2)}, m) = 1$, we have

$$\mathrm{CS}(\mathcal{H}_m^{(1)}, D_m^{(1)}, m)\mathrm{CS}(\mathcal{H}_m^{(2)}, D_m^{(2)}, m) = 1. \tag{6.3}$$

Let us first assume that $\ell \neq D_m^{(i)}$ for $i = 1, 2$; this is obviously the case if ℓ is not invariant by $\mathcal{F}$. Then, by (𝔞), we have $\ell_m^{(i)} \neq L_\infty$ for $i = 1, 2$. Therefore, each of the $\mathscr{H}_m^{(i)}$ is conjugate to one of the five pre-foliations $\mathscr{H}_j^3$, $j = 2, \ldots, 6$, so that each of the $\mathcal{H}_m^{(i)}$ is conjugate to one of the three foliations $\mathcal{H}_1^2$, $\mathcal{H}_2^2(0, 0)$, $\mathcal{H}_3^2(-2)$ (Corollary 4.5). Consulting Table 1 and using equality (6.3) as well as relations (𝔟) and (𝔠), we see that

$$\begin{aligned}&\mathrm{CS}(\mathcal{H}_m^{(1)}, D_m^{(1)}, m) = \mathrm{CS}(\mathcal{H}_m^{(2)}, D_m^{(2)}, m) = -1, \#\,(\Sigma^1 \cap D_m^{(1)}) = \#\,(\Sigma^1 \cap D_m^{(2)}) = 2,\\ &\Sigma^2 \cap D_m^{(1)} = \Sigma^2 \cap D_m^{(2)} = \{m\}.\end{aligned} \tag{6.4}$$

Let us now assume that the line ℓ is equal to one of the lines $D_m^{(i)}$, say $\ell = D_m^{(2)}$. Let us show that equalities (6.4) still hold. Since $\ell \neq D_m^{(1)}$, $\mathcal{H}_m^{(1)}$ is conjugate to one of the foliations $\mathcal{H}_1^2$, $\mathcal{H}_2^2(0, 0)$, $\mathcal{H}_3^2(-2)$. Moreover $\Sigma^2 \cap D_m^{(1)} = \{m\}$; indeed, if $\Sigma^2 \cap D_m^{(1)}$ contained another point $m' \neq m$, we would have $\ell \neq D_{m'}^{(i)}$ for $i = 1, 2$, so that $\{m'\} = \Sigma^2 \cap D_{m'}^{(i)} = \Sigma^2 \cap D_m^{(1)} \supset \{m, m'\}$, which is impossible. From Table 1, we deduce that

$$\mathrm{CS}(\mathcal{H}_m^{(1)}, D_m^{(1)}, m) = \mathrm{CS}(\mathcal{H}_m^{(2)}, \ell, m) = -1 \qquad \text{and} \qquad \#(\Sigma^1 \cap D_m^{(1)}) = 2,$$

hence

$$\mathrm{CS}(\mathcal{F}, D_m^{(1)}, m) = \mathrm{CS}(\mathcal{F}, \ell, m) = -1.$$

Since these equalities are valid for any choice of $m \in \Sigma^2 \cap \ell$ and since every line of $\mathbb{P}^2_{\mathbb{C}}$ cannot contain more than $\deg \mathcal{F} + 1 = 3$ singular points of $\mathcal{F}$, the Camacho-Sad formula (*see* [10]) $\sum_{s\in\mathrm{Sing}\mathcal{F}\cap\ell} \mathrm{CS}(\mathcal{F}, \ell, s) = 1$ implies that

$$\#\,(\Sigma^1 \cap \ell) = 2 \qquad \text{and} \qquad \Sigma^2 \cap \ell = \{m\}.$$

Equalities (6.4) are thus established in all cases. It follows in particular that $\mathrm{BB}(\mathcal{F}, m) = 0$. The point $m \in \Sigma^2$ being arbitrary, Σ^2 consists of $s \in \mathrm{Sing}\mathcal{F}$ such that $\mathrm{BB}(\mathcal{F}, s) = 0$. System (6.2) then rewrites as $\kappa_1 + \kappa_2 = 7$ and $4\kappa_1 = 16$, whose unique solution is $(\kappa_1, \kappa_2) = (4, 3)$, that is $\mathrm{Sing}\mathcal{F} = \Sigma^1 \cup \Sigma^2$, $\#\,\Sigma^1 = 4$ and $\#\,\Sigma^2 = 3$. Since $\Sigma^2 \cap (D_m^{(1)} \cup D_m^{(2)}) = \{m\}$, $\mathcal{F}$

has $3 \cdot 2 = 6$ invariant lines, which means that $\mathcal{F}$ is reduced convex. It then follows from the classification of convex foliations of degree two (*cf.* [11, Proposition 7.4] or [5, Theorem A]) that $\mathcal{F}$ is linearly conjugate to the FERMAT foliation $\mathcal{F}_0^2$. We conclude by noting that if the line ℓ is not invariant by $\mathcal{F}$, the flatness of Leg$\mathscr{F}$ and Proposition F imply that ℓ must join two non-radial singularities of $\mathcal{F}$. □

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