Nom:............................ Prénom:..........................

Exercice 1 : 6.5

A) $y(t) = x(t) + \alpha\, x(t-\theta)$ α et θ cstes $\begin{cases} \mu x(t) = cste = 0 \\ Rx(t,\tau) = Rx(\tau) \end{cases}$

1°/ $E\{y(t)\} = E\{x(t)\} + \alpha\, E\{x(t-\theta)\}$
$= 0 = cste$ 0.5

SSL $Rx(t,\tau) = E\{y(t)\,y(t-\tau)\} = E\{[x(t)+\alpha x(t-\theta)][x(t-\tau)+\alpha x(t-\tau-\theta)]\}$
$= (1+\alpha^2)\, Rx(\tau) + \alpha\,[Rx(\tau-\theta) + Rx(\tau+\theta)] = f(\tau)$ 1

2°/ $Py = \sigma y^2 + \mu y^2 = Ry(0) = (1+\alpha^2)\,Ry(0) + 2\alpha\, Rx(\theta)$ 1 x réel

3°/ $Sy(f) = TF\{Sy(\tau)\} = (1+\alpha^2)\,Sx(f) + \alpha(e^{-2\pi j f\theta} + e^{2\pi j f\theta})\,Sx(f)$
$= (1+\alpha^2 + 2\alpha\cos 2\pi f\theta)\, Sx(f)$ 1

4°/ $Sy(f) = |H(f)|^2 \cdot Sx(f) \Rightarrow |H(f)|^2 = 1+\alpha^2 + 2\alpha\cos 2\pi f\theta > 0$ 1

5°/ $H(f) = \sqrt{1+\alpha^2 + 2\alpha\cos 2\pi f\theta}$ 0.5

B) Realisation H.B.I   Correlation : D.G.E   DSP : A.C.I   1.5

Exercice 2 5

$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) + x(n)$ $x(n)$ bb décorrelé $\sigma x^2 = 1$ $\begin{cases} Rx(k) = \sigma_x^2\,\delta(\tau) \\ \mu x(n) = 0 \end{cases}$

1°/ $\mu y(n)\,[1 + a_1 + a_2] = \mu x(n) = 0 \Rightarrow \mu y(n) = 0$ 0.5

2°/ $x(n)$ SSL + S.L.I.T $\Rightarrow x(n)$ SSL 1

3°/ $Ry(k) = -a_1\, Ry(k-1) - a_2\, Ry(k-2) + \underbrace{Rx(k) * h(k)}_{=0 \quad k \neq 1}$ 1

4°/ $\begin{bmatrix} Ry(0) & Ry(1) & Ry(2) \\ Ry(1) & Ry(0) & Ry(1) \\ Ry(2) & Ry(1) & Ry(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} a_1 = \\ a_2 = \end{cases}$ 1.5

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Exercice 3 4.5

$y(n) = x(n) + \alpha\, x(n-1) + b(n)$ $b(n)$ b.b décorrelé $\begin{cases} \mu b = 0 \\ R_b(k) = \sigma^2 \delta(k) \\ \text{indep. de } x \end{cases}$

1°/ $\mu x = \sum xi\ prob(xi) = 1 \times \frac{1}{2} - 1 \cdot \frac{1}{2} = 0$ 0.5

$\sigma x^2 = \sum (xi - \mu x)^2\, prob(xi) = 1^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{2} = 1$ 0.5

2°/ $E\{y(n)\,y(n-k)\} = E\{[x(n) + \alpha x(n-1) + b(n)][x(n-k) + \alpha x(n-k-1) + b(n-k)]\}$
$= (1+\alpha^2)\, Rx(k) + \alpha\,[Rx(k+1) + Rx(k-1)] + Rb(k)$ 1

3°/ $E\{x(n)\,y(n-k)\} = E\{x(n)\,x(n-k) + \alpha\, x(n)\,x(n-1-k) + \alpha\, x(n)\, b(n-k)\}$
$= Rx(k) + \alpha\, Rx(k+1)$ 1

$$\begin{bmatrix} 1+\alpha^2+\sigma^2 & \alpha & 0 & 0 \\ \alpha & 1+\alpha^2+\sigma^2 & \alpha & 0 \\ 0 & \alpha & 1+\alpha^2+\sigma^2 & \alpha \\ 0 & 0 & \alpha & 1+\alpha^2+\sigma^2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha \\ 0 \\ 0 \end{bmatrix}$$ 1.5

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Exercice 4

$$p(x;\theta)=\begin{cases}\frac{1}{\theta}e^{-\frac{(x-a)}{\theta}} & \text{si } x\geq a\\ 0 & \text{ailleurs}\end{cases} \qquad \begin{cases}E\{x\}=a+\theta\\ \sigma_x^2=\theta^2\end{cases}$$ 1

1) $L(p(x;\theta)) = \prod_{1}^{n} p(x_i;\theta)$

$$= \frac{1}{\theta^n} e^{-\frac{1}{\theta}((x_1-a)+(x_2-a)+\dots(x_n-a))}$$

$$\ln(L(p(x;\theta))) = -\ln(\theta^n) - \frac{1}{\theta}\sum(x_i-a) = -n\ln\theta - \frac{1}{\theta}\sum(x_i-a)$$

$$\frac{d\ln(L(p(x;\theta)))}{d\theta} = -\frac{n}{\theta}+\frac{1}{\theta^2}\sum(x_i-a)=0$$

$$\Rightarrow -n\theta+\sum(x_i-a)=0$$

$$\Rightarrow \hat{\theta}=\frac{1}{n}\sum x_i - a$$ 1.5

2) $b_{\hat{\theta}} = E\{\hat{\theta}\}-\theta = E\{\frac{1}{n}\sum x_i\}-a-\theta = \frac{1}{n}\sum(a+\theta)-a-\theta = 0$ 0.5

3) $\sigma^2_{\hat{\theta}} = E\{(\hat{\theta}-E\{\hat{\theta}\})^2\} = E\{(\frac{1}{n}\sum x_i - a-\theta)^2\} = E\{[\frac{1}{n}\sum x_i - \frac{1}{n}\sum(a+\theta)]^2\}$

$$= E\{\frac{1}{n^2}[\sum(x_i-(a+\theta))]^2\}$$

puisque $\begin{cases}\text{les } x_i \text{ sont indépendants}\\ E\{x_i-(a+\theta)\}=E\{x_i-\mu_{x_i}\}=0\end{cases}$

$$\Rightarrow \sigma^2_{\hat{\theta}} = \frac{1}{n^2}\sum \underbrace{E\{[x_i-(a+\theta)]^2\}}_{\sigma_x^2} = \frac{1}{n^2}\sum\sigma_x^2 = \frac{\sigma_x^2}{n} = \frac{\theta^2}{n}$$

$\Rightarrow$ consistent ($b=0$, $\sigma \xrightarrow[n\to\infty]{} 0$) 1