Exercice 1

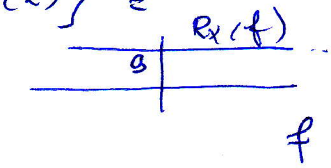
On considère l'équation aux différences suivante $y(n) = \frac{1}{3}x(n) - \frac{2}{3}x(n-2)$ tels que Rx(k)=9 δ(k)

1. Déterminer les moments de premier ordre de l'entrée x(n).
2. Tracer Sx(f).
3. On suppose que l'on simule x(n) sous matlab, tracer alors Rx(k) et Sx(f).
4. Calculer les moments d'ordre 1 de y(n).
5. Calculer et tracer Ry(k). Retrouver du graphe les moments d'ordre 1.
6. Sachant que la TFTD{x(n)}=$\sum_{-\infty}^{\infty} x(n)e^{-2\pi jfnTe}$, calculer Sy(f).

1° - $Rx(k) = 9\,\delta(k)$ bb de variance 9 et de moyenne 0

$\Rightarrow \begin{cases} \sigma_x^2 = 9 \\ \mu_x = 0 \end{cases}$

2° - $Sx(f) = TFTD\{Rx(k)\} = 9$



4° - $\mu y(n) = E\{y(n)\} = \frac{1}{3}\mu x - \frac{2}{3}\mu x = 0 = \mu y$

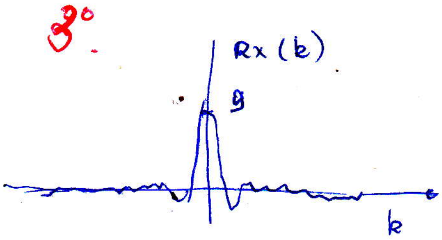
$\sigma_y^2(n) = E\{y(n)^2\} - \cancel{\mu y}^2 = E\{(\frac{1}{3}x(n) - \frac{2}{3}x(n-2))^2\}$

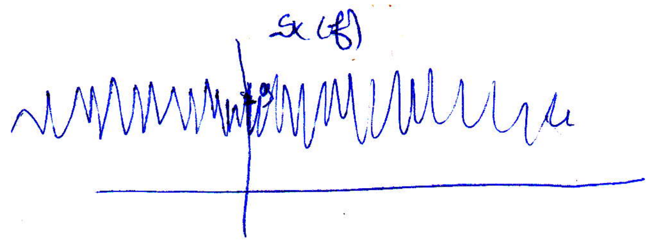
$= \frac{1}{9}E\{x(n)^2\} + \frac{4}{9}E\{x(n-2)^2\} - \frac{4}{3}E\{x(n)\,x(n-2)\}$

$= \frac{1}{9}(\sigma_x^2 + \cancel{\mu_x^2}) + \frac{4}{9}(\sigma_x^2 + \cancel{\mu_x^2}) - \frac{4}{3}Rx(2)$

$= \frac{1}{9}Rx(0) + \frac{4}{9}Rx(0) - \frac{4}{3}\times 0 = \frac{5}{9}Rx(0) = \frac{5\times\cancel{9}}{\cancel{9}} = 5$

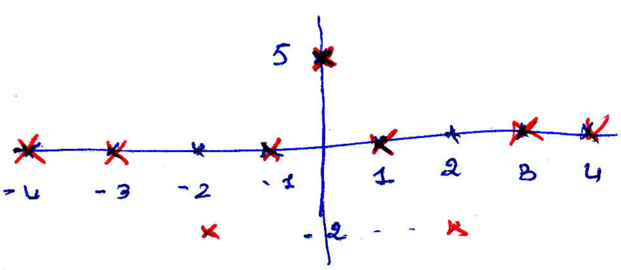
3° -





5° - $Ry(k) = E\{x(n)\,x(n-k)\}$ modèle MA $(b_0 = 1/3 \quad b_1 = 0 \quad b_2 = -2/3)$

$\Rightarrow \begin{cases} Ry(0) = \sigma^2\left(\frac{1}{3}^2 + 0^2 + \left(-\frac{2}{3}\right)^2\right) = \sigma^2 \cdot \frac{5}{9} = 5 \\ Ry(1) = Ry(-1) = \sigma^2\left(\frac{1}{3}\times 0 + 0\times -\frac{2}{3}\right) = 0 \\ Ry(2) = Ry(-2) = \sigma^2\left(\frac{1}{3}\times -\frac{2}{3}\right) = -2 \\ Ry(k \geq 3) = Ry(k \leq -3) = 0 \end{cases}$



$\Rightarrow \begin{cases} \mu y = 0 \\ \sigma y^2 = 5 \end{cases}$

6° - $Sy(f) = TFTD\{Ry(k)\}$

$= -2e^{+4\pi jfnTe} + 5 - 2e^{-4\pi jfnTe}$

$= 5 + 4\cos(4\pi fnTe)$

Exercice 2

Soient (x_1, x_2, ...x_N) des échantillons indépendants correspondant à une population dans la moyenne est μ et de variance σ^2 . On considère les 2 estimateurs suivants de la moyenne:

$$\widehat{\mu_1} = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad \widehat{\mu_2} = ax_1 + (1-a)x_2 \text{ où } 0 \le a \le 1$$

1. Calculer le biais et la variance de chaque estimateur.
2. Etudier leur consistance.
3. Lequel vous semble préférable (Justifier) ?

1°- $\hat{\mu}_1 = \frac{1}{N}\sum_{i=1}^{N} x_i$

$b_{\hat{\mu}_1} = E\{\hat{\mu}_1\} - \mu = E\{\frac{1}{N}\sum_{i=1}^{N} x_i\} - \mu = \frac{1}{N}\sum_{i=1}^{N} E\{x_i\} - \mu$

$= \frac{1}{N}\sum_{i=1}^{N}\mu - \mu = \frac{1}{N}.N.\mu - \mu = 0$

- $\sigma^2_{\hat{\mu}_1} = E\{(\hat{\mu}_1 - E\{\hat{\mu}_1\})^2\} = E\{(\frac{1}{N}\sum_{i=1}^{N} x_i - E\{\frac{1}{N}\sum_{i=1}^{N} x_i\})^2\}$

$= E\{\frac{1}{N^2}[\sum_{i=1}^{N}(x_i - E\{x_i\})]^2\}$

Sachant que les x_i sont indépendants et que $E\{x_i - E\{x_i\}\} = 0$

alors $\sigma^2_{\hat{\mu}_1} = E\{\frac{1}{N^2}[\sum_{i=1}^{N}(x_i - E\{x_i\})]^2\} = \frac{1}{N^2}\sum_{i=1}^{N}\underbrace{E\{(x_i - E\{x_i\})^2\}}_{\sigma^2}$.

$= \frac{1}{N^2}\sum_{i=1}^{N}\sigma^2 = \frac{1}{N^2}.N.\sigma^2 = \sigma^2/N$

- $\hat{\mu}_2 = ax_1 + (1-a)x_2$

$b_{\hat{\mu}_2} = E\{\hat{\mu}_2\} - \mu = E\{ax_1 + (1-a)x_2\} - \mu = a\,E\{x_1\} + (1-a)E\{x_2\} - \mu$

$= a\mu + (1-a)\mu - \mu = a\mu + \mu - a\mu - \mu = 0$

- $\sigma^2_{\hat{\mu}_2} = E\{(\hat{\mu}_2 - E\{\hat{\mu}_2\})^2\} = E\{(ax_1 + (1-a)x_2 + E\{ax_1 + (1-a)x_2\})^2\}$

$= E\{[a(x_1 - E\{x_1\}) + (1-a)(x_2 - E\{x_2\})]^2\}$

$= E\{(a(x_1 - E\{x_1\}))^2 + ((1-a)(x_2 - E\{x_2\}))^2 + 2a(1-a)(x_1 - E\{x_1\})(x_2 - E\{x_2\})\}$

$= a^2 \underbrace{E\{(x_1 - E\{x_1\})^2\}}_{\sigma^2} + (1-a)^2 \underbrace{E\{(x_2 - E\{x_2\})^2\}}_{\sigma^2} + 2a(1-a)\cancel{E\{(x_1 - E\{x_1\})(x_2 - E\{x_2\})\}}$

Sachant que x_1 et x_2 sont indépendants $\Rightarrow E\{(x_1 - E\{x_1\})(x_2 - E\{x_2\})\} = 0$

$\Rightarrow \sigma^2_{\hat{\mu}_2} = [a^2 + (1-a)^2]\,\sigma^2$

2°- $b_{\hat{\mu}_1} = 0$ et $\lim_{N \to \infty} \sigma^2_{\hat{\mu}_1} = \sigma^2/N = 0 \Rightarrow \mu_1$ consistant

$b_{\hat{\mu}_2} = 0$ et $\lim_{N \to \infty} \sigma^2_{\hat{\mu}_2} = a^2 + (1-a)^2\,\sigma^2 \neq 0 \Rightarrow \mu_2$ non consistant

3°- le premier car il est consistant.

Exercice 3

On considère le processus y(t) tel que $y(t) = \frac{1}{2T}\int_{t-T}^{t+T} x(\tau)d\tau$ où x(t) est un signal aléatoire SSL tel que $S_x(f) = \frac{3}{9+4\pi^2 f^2} + 4\delta(f)$

1. Déterminer les moments d'ordre1 du signal d'entrée.
2. Déterminer E{y(t)} et Sy(f).
3. On considère maintenant que l'entrée est un bruit blanc réduit, déterminer $R_y(\tau)$.
4. On considère que le bruit blanc est Gaussien, tracer l'entrée et la sortie.

$\cos(a+b) = \cos a \cos b - \sin a \sin b$ $\quad \sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b))$ $\quad \sin a \sin b = -\frac{1}{2}(\cos(a+b) - \cos(a-b))$

$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$ $\quad \tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$

$TF\{\delta(t-t_0)\} = e^{-2\pi j f t_0}$ $\quad TF\left\{A \prod_\theta (t)\right\} = A\theta \sin c(f\theta)$

$TF\left\{A \bigwedge_\theta (t)\right\} = A\theta \sin c^2(f\theta)$ $\quad TF\left\{e^{-at}U(t)\right\} = \frac{a}{a + 2\pi j f}$

$TF\left\{e^{-a|t|}\right\} = \frac{2a}{a^2 + 4\pi^2 f^2}$

1° $S_x(f) = \frac{3}{9^2 + 4\pi^2 f^2} + 4\delta(f) \Rightarrow R_x(\tau) = TF^{-1}\{S_x(f)\} = \frac{1}{2} e^{-3|\tau|} + 4$

$\begin{cases} R_x(0) = \frac{1}{2} + 4 = \mu_x^2 + \sigma_x^2 \\ R_x(\infty) = 4 = \mu_x^2 \end{cases} \Rightarrow \begin{cases} \mu_x = 2 \\ \sigma_x^2 = 1/2 \end{cases}$

2° $x(t)$ SSL et $y(t) = \frac{1}{2T}\int_{t-T}^{t+T} x(\tau) \Rightarrow h(t)$ SLIT $\Rightarrow y(t)$ SSL

$\Rightarrow \begin{cases} \mu_y = \mu_x \cdot H(0) \\ S_y(f) = |H(f)|^2 \cdot S_x(f) \end{cases}$

$y(t) = \frac{1}{2T}\int_{t-T}^{t+T} x(\tau)\, d\tau = \frac{1}{2T}\int_{-\infty}^{+\infty} \prod_{2T}(t-\tau)\, x(\tau)\, d\tau \Rightarrow h(t-\tau) = \frac{1}{2T}\prod_{2T}(t-\tau)$

$\Rightarrow h(t) = \frac{1}{2T}\prod(t) \Rightarrow H(f) = \frac{1}{\cancel{2T}} \cdot \cancel{2T} \operatorname{sinc}(2fT) = \operatorname{sinc}(2fT)$

* $\mu_y = \mu_x \cdot H(0) = 2 \cdot \operatorname{sinc}(2 \cdot 0 \cdot T) = 2$

* $S_y(f) = |H(f)|^2 \cdot S_x(f) = \operatorname{sinc}^2(2fT) \cdot \left[\frac{3}{9 + 4\pi^2 f^2} + 4\delta(f)\right]$

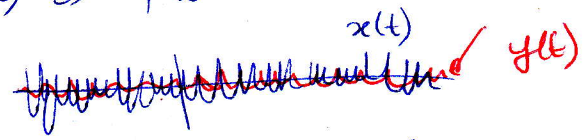
3° Entrée bb reduit $\Rightarrow R_x(\tau) = \sigma^2 \delta(\tau)$ avec $\sigma^2 = 1$

$S_y(f) = |H(f)|^2 \cdot S_x(f) = \operatorname{sinc}^2(2fT) \cdot TF\{R_x(\tau)\} = \sigma^2 \operatorname{sinc}^2(2fT) = \operatorname{sinc}^2(2fT)$

$R_y(\tau) = TF^{-1}\{S_y(f)\} = \frac{1}{2T} TF^{-1}\{\operatorname{sinc}^2(2fT)\} = \frac{1}{2T} \bigwedge_{2T}(\tau)$

4° $R_x(\tau) = \sigma^2 \delta(\tau) \Rightarrow \mu_x = 0$

$\Rightarrow x(t)$ Gaussien



$h(t)$: filtre moyenneur $\Rightarrow$ passe-bas