Advanced Signal Processing

(Course materials)

Master 1 : Telecommunications & Networks

Faculty of Electrical Engineering

Houari Boumediene University of Science and Technology

Personal website: http://perso.usthb.dz/~akourgli/

e-mails: akourgli@usthb.dz , kourgliassia@gmail.com

Course content

Introduction: Miscellaneous Reminders

Chapter 1. Reminders on digital filters (RIF and RII) (3 Weeks)

1. Transformed in Z
2. Structures , transfer functions , stability and implementation of digital filters (RIF and RII)
3. Digital minimum phase filter
4. Synthesis methods of FIR filters and IIR filters
5. Multirate digital filters

Practical work n°1: Synthesis of RIF and RII filters

Practical work n°2: Multi -rate filtering

Chapter 2. Random Signals and Stochastic Processes (4 Weeks)

1. Reminder about Random Processes
2. Notions of stochastic processes
3. Stationarities in the broad and strict sense and Ergodicity
4. Examples of stochastic processes (Poisson process, Gaussian and Markovian process)
5. Higher order statistics (Moments and cumulants , Polyspectra , non-Gaussian processes, nonlinear treatments)
6. Power Spectral Density
7. Matched filter , Wiener filter
8. Periodogram , correlogram, averaged periodogram , smoothed periodogram
9. Introduction to Particle Filtering

Practical work n°3: Random Processes, periodogram and variants

Practical work n°4: Matched filtering and Wiener filtering

Chapter 3. Adaptive Digital Filtering (4 Weeks)

1. Parametric Methods
2. AR model (Lévinson , Yulewalker , Burg, Pisarenko , Music …)- ARMA model
3. LMS Stochastic Gradient Algorithm

4. RLS Recursive Least Squares Algorithm

Practical work n°5: AR modeling and adaptive filtering (LMS)

Chapter 4. Time-Frequency and Time-Scale Analysis (4 Weeks)

1. Time-frequency duality
2. Short term Fourier transform
3. Continuous, discrete and dyadic wavelets
4. Multi-resolution analysis and wavelet bases
5. Wigner-Ville transform
6. Time-Scale Analysis ,

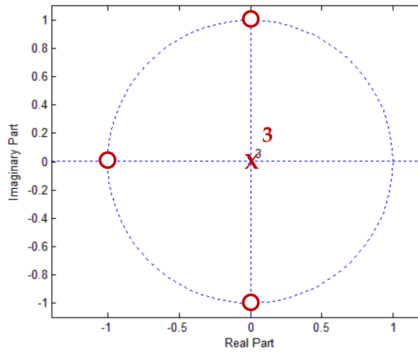
Practical work n°6: Time- frequency /Scale transforms

Exercices Set n°1: Synthe Filters and Multi-cadence Filtering

1. Let $H(z)$ be the transfer function of a causal SLIT with: $H(z) = \dfrac{az-1}{z-a}$ with a real.

Determine the values of a for which $H(z)$ corresponds to a stable system. Take a value of a =0.5. Then represent the poles and zeros of the function, the region of convergence. Give and Trace $|H(f)|$. Is this filter minimum phase?

2. Suppose that the plot of the poles and zeros of this system is as follows:



- Is it an RIF or RII filter? (justify your answer)
- Give the approximate shape of H(f)
- Determine H(z) then determine and plot h(n)
- From h(n), study the stability, the causality and the invariance of this filter.
- Calculate and plot H(f) for at least 3 values
- Does this filter have a constant group delay (justify)
- Determine its response for a step input x(n)=U(n).

3. We consider that the recurrence equation of the following system is given as follows:

$y(n)= x(n) - x(n-2) - 0.878\, y(n-2)$

- Study the causality and the invariance of this system
- Is it an RIF or RII filter?
- Determine H(z) and plot the poles and zeros, deduce the role of this filter:
- Give the approximate shapes of h(n) and |H(f)|
- Determine the impulse response h(n) and plot it for the first 3 values
- We assume that fe= 6 kHz, what will be the output of the filter if we give as input: a signal noisy with a sinusoid of 500 Hz then a signal composed of 2 sinusoids, one of 1500 Hz and the other of 2500Hz

4. Obtain the coefficients of a low-pass FIR filter by the windowing method to obtain the following specifications:

- Ideal cutoff frequency: fc=1.75 kHz
- Transition width: Δf=0.5 kHz
- Attenuation in attenuated band: $A=-20\log_{10}(\delta) > 51$ dB with $\delta=\min(\delta_1, \delta_2)$
- Sampling frequency: fe=8 kHz

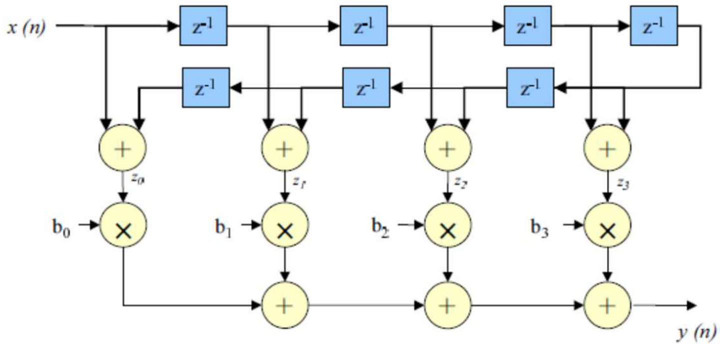
5. We wish to approach an ideal high-pass filter by a finite impulse response filter, synthesized by the windowing method. This filter must meet the following specifications:

- ✓ Cutoff frequency f_c=2 kHz
- ✓ Transition width: Δf=0.5 kHz
- ✓ Attenuation in attenuated band: $A=-20\log_{10}(\delta) > 40$ dB with $\delta=\min(\delta_1, \delta_2)$
- ✓ Sampling frequency: f_e=8 kHz

- Determine the exact mathematical expression for h'(n).
- Calculate h'(0) and approximately plot h'(n).
- What is the advantage of windowing and what is its disadvantage?
- What is the disadvantage of this filter synthesis technique?
- List an advantage and a disadvantage of synthesis by RIF filters as well as by IIRs.

6. We consider an FIR filter of length N =8 whose Structure is given opposite:
With b_0=0.2, b_1=0.3, b_2=0.4, b_3=0.5

- Determine the difference equation
- Determine the transfer function then h(n)
- Then identify the type of RIF filter (I, II, III or IV)
- What is its disadvantage?



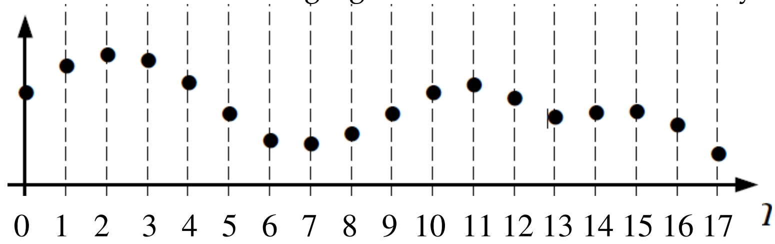
7. We consider the normalized single-pole analog low-pass filter $H_N(p)=1/(p+1)$. It is considered that the cutoff frequency normalized to -3db occurs for $f_c=f_e/10$. By bilinear transformation, find the equivalent low-pass digital filter H(z).

8. Consider the following denormalized low-pass transfer function H(p): $(p+0.1)/((p+0.1)^2+\omega_a^2)$. Knowing that $\omega_a=4$, use the bilinear transform method to transform this filter into a digital one. The digital filter will have its resonance for $\omega_N=\pi/(2.T_e)$.

9 . Let the signal x(n)= n U(n), give and plot the resulting signal if:

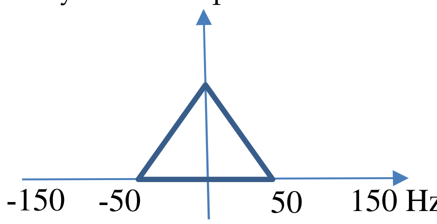
- We decimate it by 2, by 3 and by 4
- We interpolate it by 2 , by 4
- We decimate it by 2 then we interpolate it by 2 and if we reverse the 2 operations
- We decimate it by 3 and we interpolate it by 2 and if we reverse the 2 operations
  Check that decimation is not invariant to translation contrary to interpolation;

10 . Consider the following signal, we want to decimate it by 2 then interpolate it by 2 as well

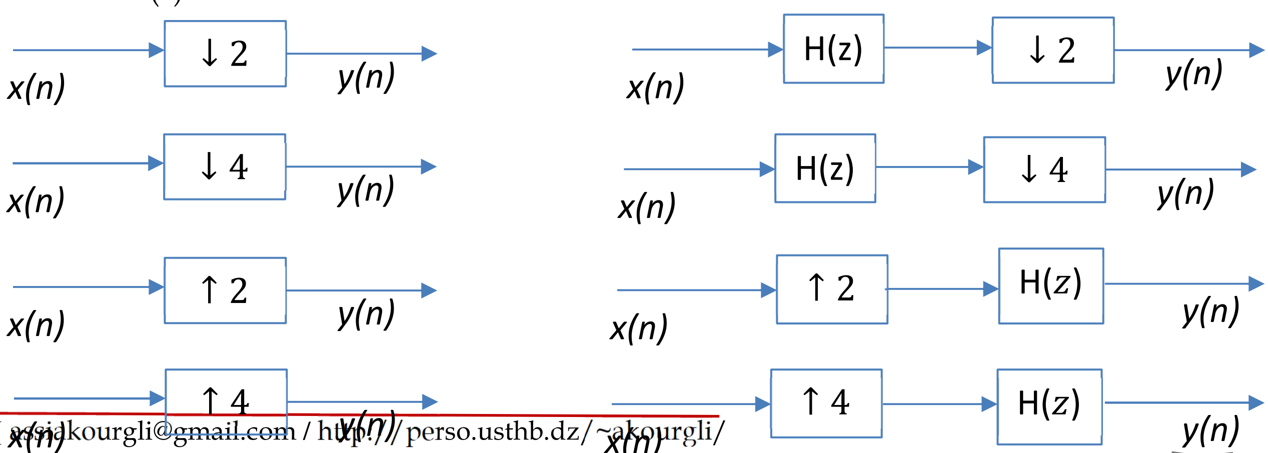


- Give the plot of the decimated then interpolated signal
- We consider that the frequency of the original signal is 10kHz, give the frequency at the output of the decimator then that of the interpolator.

11 . We consider the signal x(n) whose TFTD is given as opposite,



- Plot the TFTD of the corresponding output for the 8 cases:
- Give the cutoff frequency of the low-pass filter H(z) for each case
- Specify the role of H(z) in each case



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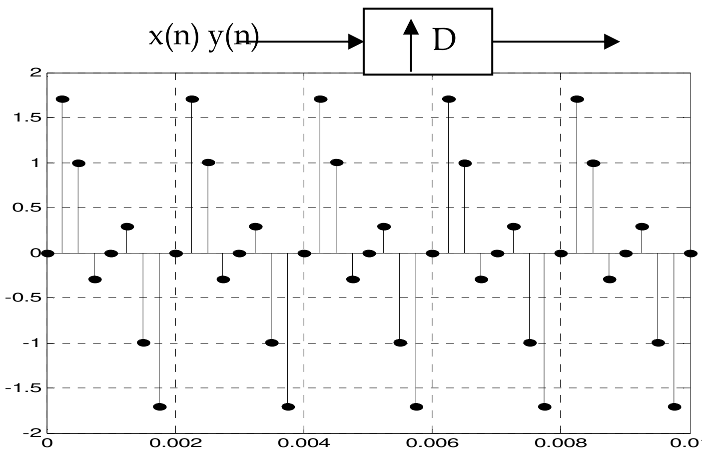
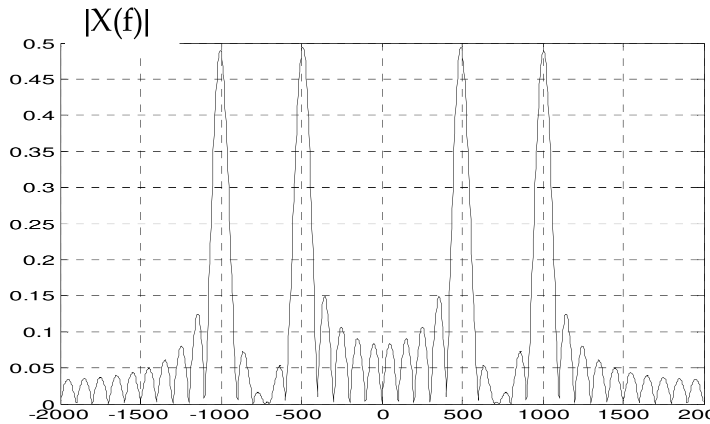
12 . Before converting a digital signal x(n) (below signal and its TFTD) , into an analog signal, we decide to interpolate it by D.





- What is the interest of interpolation in this case?
- Determine the expression of the signal x(n).
- Plot the interpolated signal for D=3 and its TFTD.
- We consider that D=6, plot again the interpolated signal and its TFTD.
- Give the initial and final polyphase decomposition for D=3
- Plot the signal and its DFT in the ca where the interpolation is followed by a low-pass filter at fe/2

13 . An audio signal is recorded was transmitted at a frequency of 30 kHz, on reception before converting it to analog, a frequency change is made such that the new frequency is 45Hz.

- Give the general scheme of this operation using decimation and interpolation
- Give the cutoff frequency of each low-pass filter, which should be kept

Solutions

1 . The module is always worth 1 (all-pass cell), Maximum of phase
2. All poles at 0 then RIF, $H(z)=z^{-3}+z^{-2}+z^{-1}+1$, h(n)= δ(n-3)+ δ(n-2)+ δ(n-1)+1,
$H(f)=2(\cos(3\pi fTe)+\cos(\pi fTe))e^{-3\pi jfTe}$, Group delay cst, x(n)=U(n) then y(n)= U(n-3)+ U(n-2)+U(n-1)+U(n)

3. Causal, stable, RII, notch filter, $H(z)=\dfrac{K(z^2-1)}{z^2+0.878}=\dfrac{K(1-z^{-2})}{1+0.878\,z^{-2}}$

4. Δf =0. 125 *fc* = 0.4375 $h(n)=f_c\dfrac{\sin(\pi n f_c)}{\pi n f_c}=\dfrac{\sin(\pi n 0.4375)}{\pi n}$ for k≠0 and $h(0)=f_c$ Hamming ⇒N=53

The filter coefficients are symmetrical, it will therefore suffice to calculate the values from h(0) to h(26).

h(0) = fc = 0.4375w(0) = 0.54 + 0.46cos(0) = 1h(0) = hN(0) w(0) = 0.4375

$$w_{Ham}(n)=\begin{cases}0.54+0.46\cos(\frac{2\pi n}{N-1}) & pour\ |n|\le\frac{N-1}{2}\\ 0 & ailleurs\end{cases}=\begin{cases}0.54+0.46\cos(\frac{2\pi n}{52}) & pour\ |n|\le 26\\ 0 & ailleurs\end{cases}$$

h(0)=0.4375, h(1)=h(-1)=0.311, h(2)=h(-2)=0.060, h(3)=h(-3)=-0.0856, h(4)=h(-4)=-0.053, h(5)=h(-5)=0.0325, h(6)=h(-6)=0.0434, h(7)=h(-7)=-0.0075 , h(8)=h(-8)=-0.0319 ,…………, h(26)=h(-26)=-0.0009.
To make the filter causal, we add 26 to each of the indices.

5 . Δf = 0. 5 f_c = 0.125 $h(n)=-f_c\dfrac{\sin(\pi n f_c)}{\pi n f_c}$ for k≠0 and $h(0)=1-f_c$ A>40 Hanning ⇒N=51

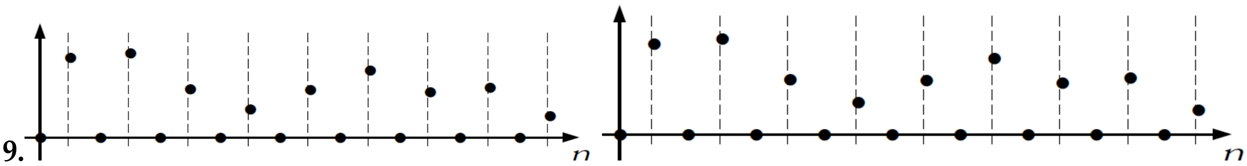
6. y(n)=b $_3$ (x(n-3)+x(n-4))+b $_2$ (x(n-2)+x(n-5))+b $_1$ (x(n-1) +x(n-6))+b $_0$ (x(n)+x(n-7))
Type II, not suitable for a high pass (a zero in -1)

7 $\cdot\ \omega_A = \frac{2}{T_e} tg\left(\frac{\pi}{10}\right) = \frac{0.65}{T_e} \Rightarrow H(p) = H_N(p)\big|_{p/\omega_A} = \frac{\omega_A}{p+\omega_A} = \frac{0.65/T_e}{p+0.65/T_e}$

$$p = \frac{2}{T_e}\frac{1-z^{-1}}{1+z^{-1}} \Rightarrow H(z) = H(p)\big|_{p=\frac{2}{T_e}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{0.65/T_e}{\frac{2}{T_e}\frac{1-z^{-1}}{1+z^{-1}}+0.65/T_e} = \frac{0.245(1+z^{-1})}{1-0.509z^{-1}}$$

8. $\omega_a = 4 = \frac{2}{T_e} tg\left(\frac{\pi}{4}\right) \Rightarrow T_e = 0.5 \Rightarrow p = 4\frac{1-z^{-1}}{1+z^{-1}} \Rightarrow H(z) = \frac{4\frac{1-z^{-1}}{1+z^{-1}}+0.1}{\left(4\frac{1-z^{-1}}{1+z^{-1}}+0.1\right)^2+16} = \frac{0.125+0.006z^{-1}-0.118z^{-2}}{1+0.0006z^{-1}-0.950z^{-2}}$

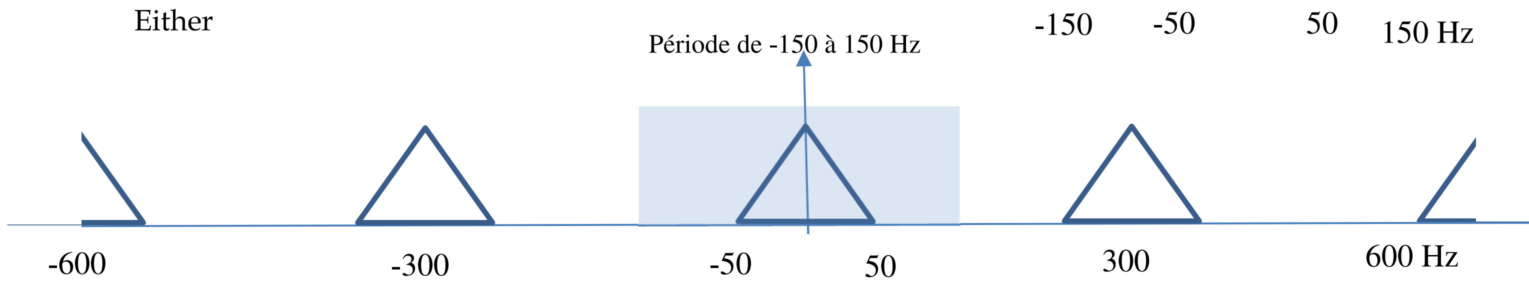
9.



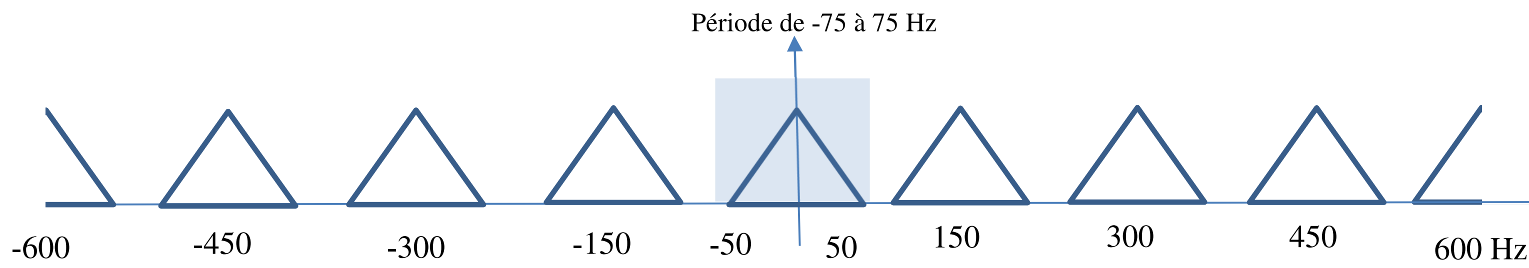
After decimation $f_e = 5kHz$ After interpolation $f_e = 10kHz$

10. The original signal is periodic with a period of 300Hz

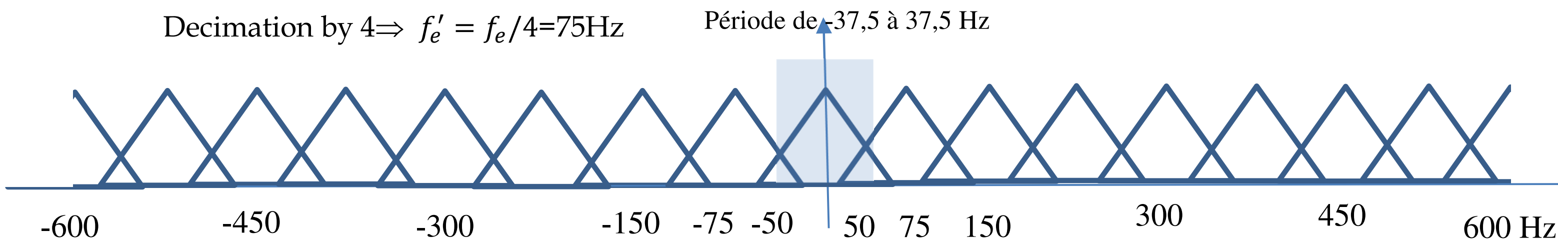
Either



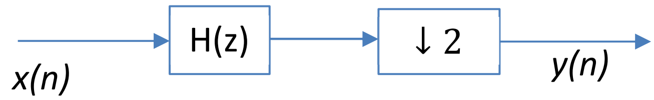
Decimation by 2$\Rightarrow$ $f_e' = f_e/2$=150Hz

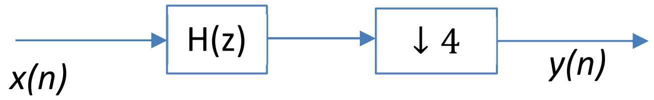


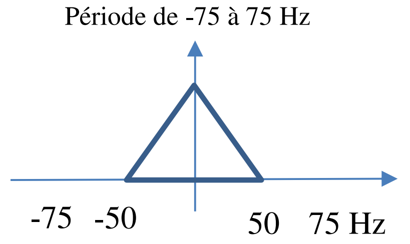
Decimation by 4$\Rightarrow$ $f_e' = f_e/4$=75Hz

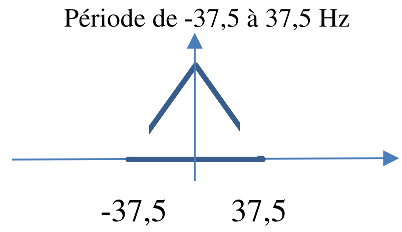


This is why we place a low pass filter at $f_e'/2 = f_e/(2M)$

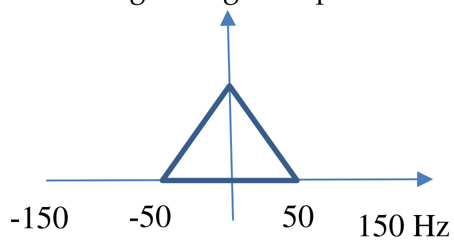


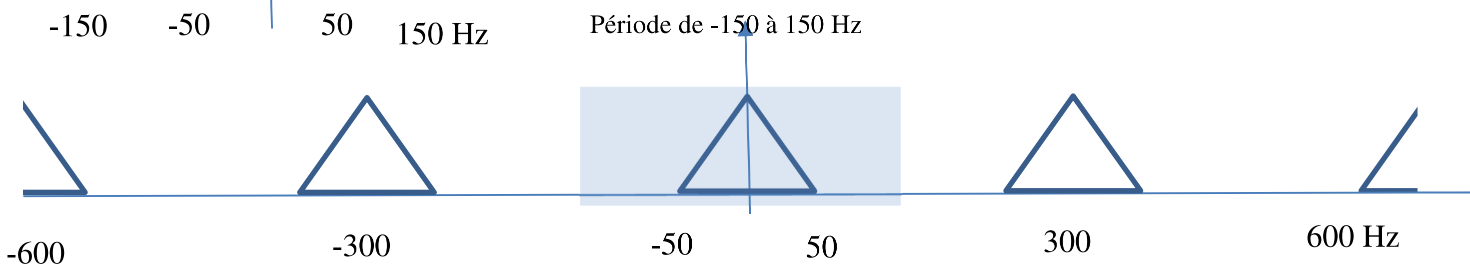




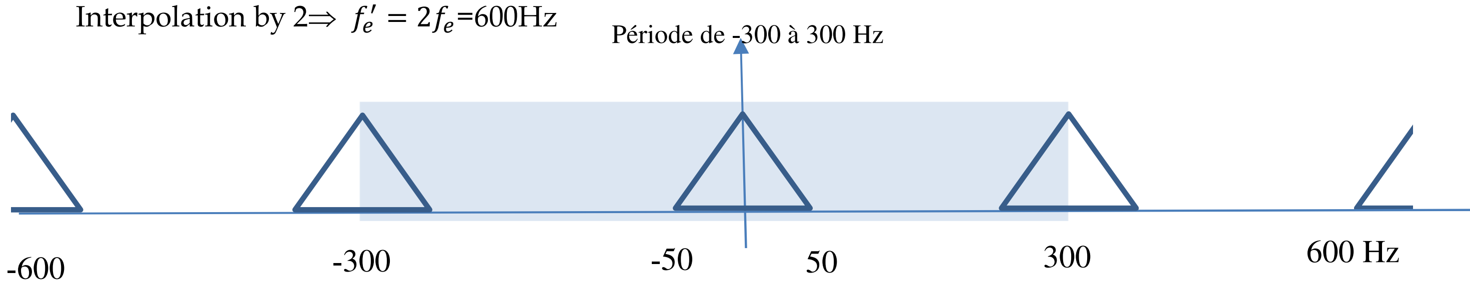


The original signal is periodic with a period of 300Hz

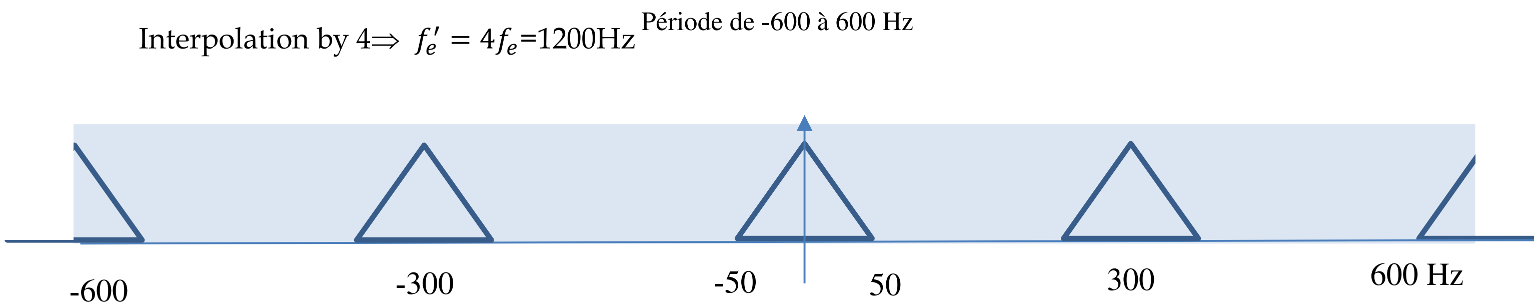




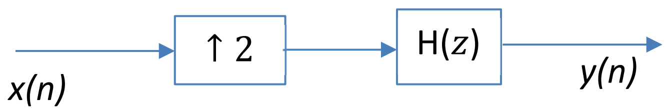
Interpolation by 2⇒ $f_e' = 2f_e$=600Hz

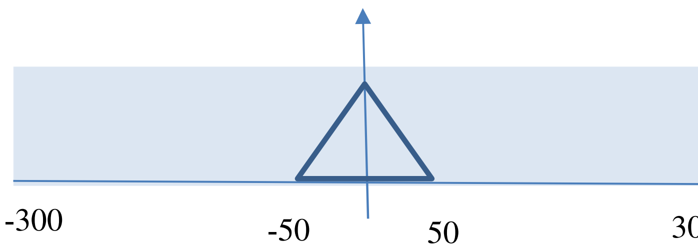


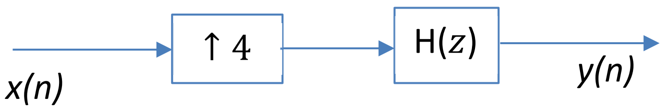
Interpolation by 4⇒ $f_e' = 4f_e$=1200Hz

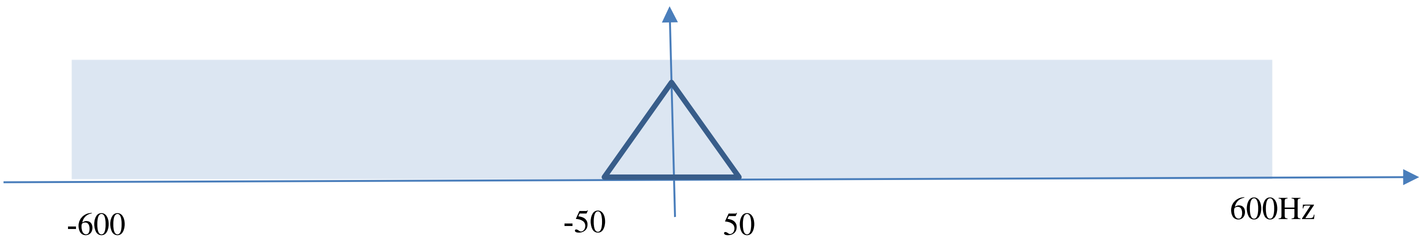


Appearance of mirror spectra ⇒Low-pass filter at $f_e/2$







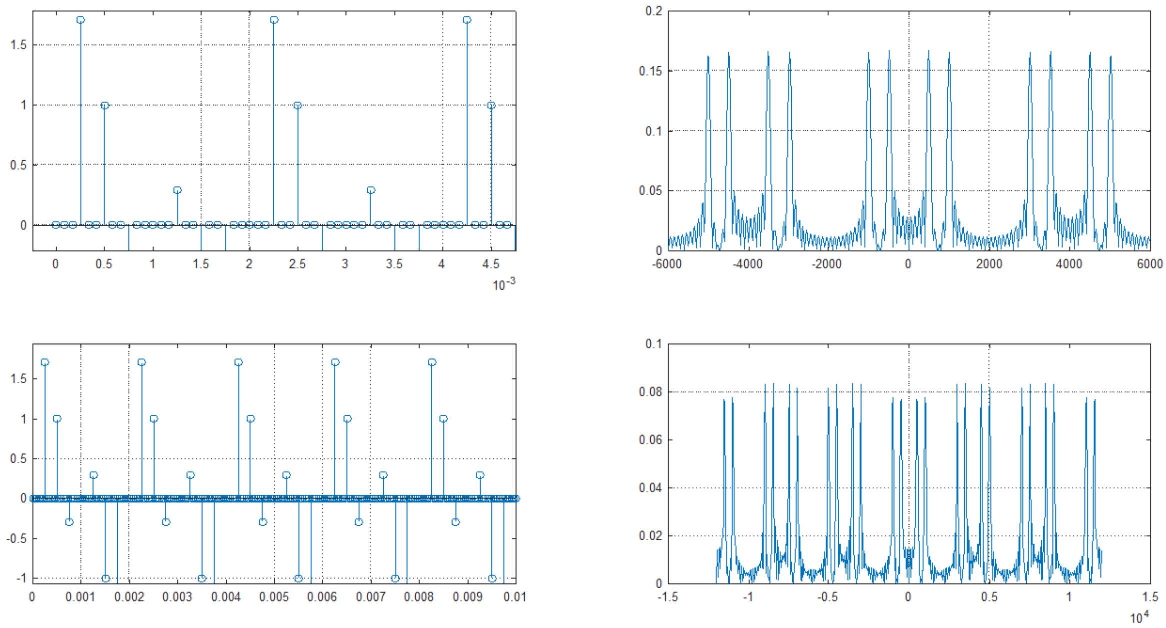


$f_c = \frac{300}{2.2} = 75, f_c = \frac{300}{2.4} = 37{,}5$ $\quad f_c = \frac{300}{2} = 150, f_c = \frac{300}{2} = 150$

Decimation: Anti-aliasing Interpolation: Anti-mirror (See course)

11. Gain Accuracy Before Digital-to-Analog Conversion

- h(n) = sin(2*pi*500*n) + sin(2*pi*1000*n) f_e *=4kHz*
- D=3 ⇒Insert 2 zeros between 2 consecutive samples. TFTD over 3 periods from –6kHz to 6kHz
- D=6 ⇒Insert 5 zeros between 2 consecutive samples. TFTD over 6 periods from –12kHz to 12kHz



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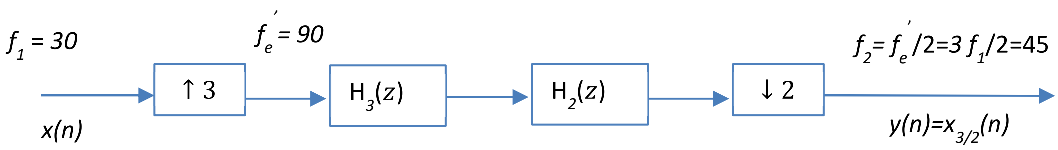
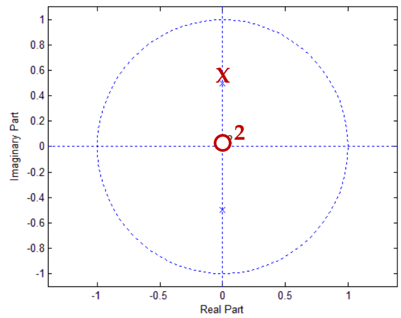


$f_c = f_1/2 = f_e'/6 = 15$ $\quad f_c = f_e'/4 = 22.5$

Additional exercises



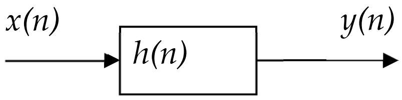
1. Suppose the plot of the poles and zeros of the following system:

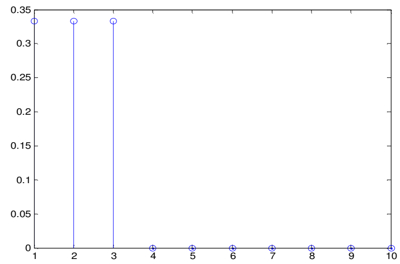
- Is it an RIF or RII filter? (justify your answer)
- Give the approximate shape of H(f)

- Determine H(z) then determine the recurrence equation
- Determine and plot h(n)
- From h(n), study the stability, the causality and the invariance of the filter

x

2. Consider the following system :

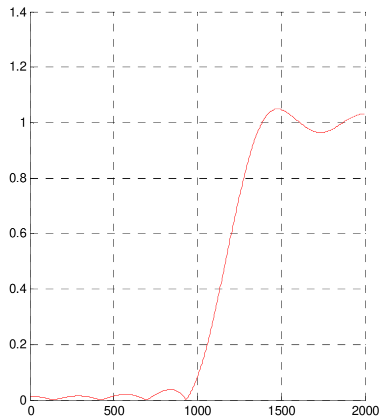


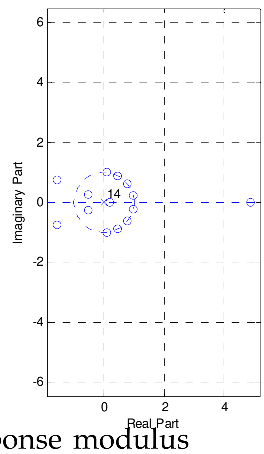


We assume that h(n) is given as opposite.
- Study causality. Is it an RIF or RII filter?
-From the expression of h(n), deduce the role of this filter:
- Determine the recurrence equation of the system:
- Determine the poles and zeros of this filter then give their plot. Deduce an approximate plot of |H(f)|
- Calculate and plot |H(f) |then deduce the plot of the modulus of the DFT for N=6.

3. During the transmission of a digital signal (sampled at a frequency of 2.5 kHz), it was affected by localized noise between the 350 Hz and 550 Hz frequency bands. We want to eliminate the noise by the use of an FIR filter having a transition band Δf=100 Hz. Design this filter using the windowing method. We want an attenuation in the attenuated band $A=-20\log_{10}(\delta) > 20$ dB.

- Plot ideal H(f)
- Determine h(n) and the order N of the filter
- Calculate h(0), h(1)=h(-1)
- Then plot approximately H(f)





4. Plots of the modulus of the frequency response from 0 to fe/2 and of the poles and zeros of a digital filter synthesized by the window method are given as follows:

- Is the phase shift of this filter linear? (Justify)
- Determine H(k) then the impulse response h(n)
- Calculate h(0).
- We want to use a hanning window. What happens to the plot of the frequency response modulus (superimpose it on H(f))

5. We consider the third-order analog filter associated with the Butterworth approximation function. Realize the corresponding low-pass digital filter using the bilinear transformation. The cutoff frequency is fc=1 kHz and the sampling frequency fe=10 kHz.

6. We seek to achieve a digital filter equivalent to the denormalized analog filter of transmittance

$$H_A(p) = \frac{1}{1+0.2p}$$

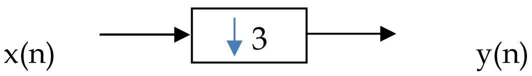
- Determine the impulse response $h_a(t)$
- Calculate the response in z of this filter obtained by bilinear transformation for a sampling frequency Te = 0.2. What is its cutoff frequency? Calculate the z-response of a similar filter with the same cutoff frequency as the analog filter.

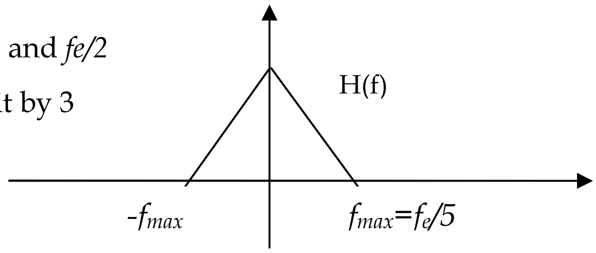
7. Consider the following signal: h(n) = sin(2*pi*150*n) + sin(2*pi*350*n) and the modulus of the DFT whose respective plots are given above: (**see exo 9**)

- Determine Te.
- What is the point of interpolation?
- Plot the signal (the first 20 values) and the TF obtained for an interpolation of 2 then of 4 (without low-pass filtering)

- Why do we use a low-pass filter during interpolation, where do we place it?
- What type of Chebeychev filter is used in this case and why?
- Give the final polyphase decomposition for an interpolation of 2 by giving the expression of the polyphase filters

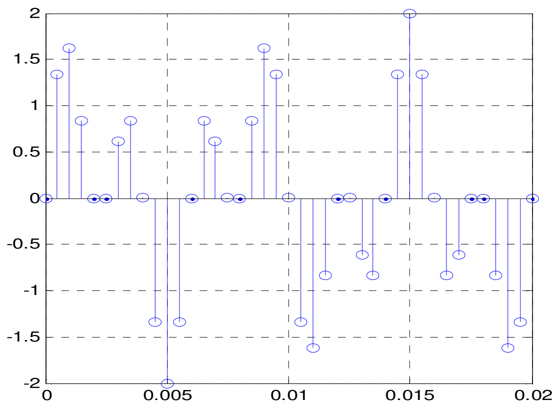
8. In order to transmit a signal x(n) (whose TFTD between – *fe/2* and *fe/2* is given opposite) more quickly, we decide to decimate it by 3

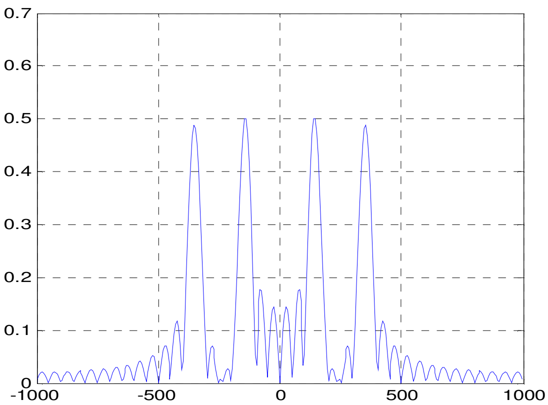




- Plot the TFTD obtained after decimation (*fe* =30kHz)
- We want to place an anti-aliasing filter, where should we place it? (justify), plot the TFTD again
- Give the initial and final polyphase decomposition by giving the expression of each filter.

9. Consider the following signal: h(n) = sin(2*pi*150*n) + sin(2*pi*350*n) and the modulus of the DFT whose respective plots are given below:





- Plot the signal and the TF obtained for a decimation of 2 then of 4 (with low-pass filtering)
- Give the final polyphase decomposition for a decimation of 4 by giving the expression of the polyphase filters
- What is the point of polyphase decomposition
- Why do we use a low pass filter during decimation?

Solutions: assessments and exams of previous years

Exercices Set No. 2 : Random processes

1. Let X(t) and Y(t) be two independent broad-sense stationary (SSL) random signals such that $R_X(\tau) = 9\, e^{-a|\tau|}$ and $R_Y(\tau) = b\,\delta(\tau) + c$. We consider Z(t) = X(t) + Y(t),

- What conditions must be imposed on the constants a, b and on c (Justify)
- Compute the first order statistics of Z(t)
- Calculate $R_Z(t, \tau)$. Is Z(t) SSL?
- Can we calculate its DSP $S_z(f)$? if so calculate the.

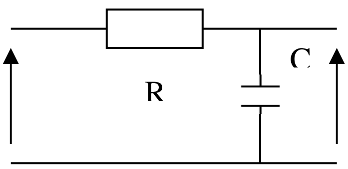
2. Let the random process be $Z(t) = X\cos(2\pi f_0 t) - Y\sin(2\pi f_0 t)$, where σ; f_0 is a constant.

a) Calculate the mean value E{Z(t)} and the variance σ_z^2 .

b) Calculate the autocorrelation function $R_z(t_1, t_2)$ and examine whether the process is SSL.

3. The input x(t) of the circuit is a white noise of autocorrelation function $R_x(\tau) = \sigma_x^2\,\delta(\tau)$

- Determine the spectral density of the output y(t), denoted $S_y(f)$
- Determine the autocorrelation function of the output y(t), denoted $R_y(\tau)$ as well as its power
- Replace R with a self L and C with a resistor and repeat the questions



4. The signals x(n) and y(n) were obtained by filtering, by means of a finite impulse response filter, a centered Gaussian white noise b(n) of variance σ^2 .

A] The filtering equation is: x(n) = 2.b(n) + 0.5.b(n -1) -0.2.b(n - 2) + 0.1.b(n - 3)

- Calculate, as a function of σ^2 , the autocorrelation coefficients of order 0,1,2,3 of the signal x(n).
- We will note Rxx (0), Rxx (1), Rxx (2), Rxx (3) for these coefficients.
- Is the distribution of the amplitude levels of the signal x(n) Gaussian (without justifying)?

B] y(n) = b(n) - b(n - 2)

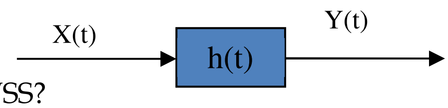
- Give the TZ of the impulse response of the filter which made it possible to obtain y(n) from b(n).
- Place the zeros of this filter on a unit circle, what are the frequency(ies) cut by this filter?
- Plot approximately its frequency response.

C] We now consider the intercorrelation coefficient Rxy(k), between the signals x(n) and y(n) given by Rxy(k) = E[x(n) y(n - k)*].

Calculate Rxy (0), Rxy (1), Rxy (2), Rxy(-1), Rxy(-2)

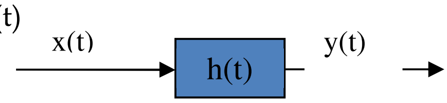
5. Let $X(t) = A + b(t)$be a real random signal, where A is a real constant and $b(t)$ is white noise with power spectral density σ_b^2, and let there be an averaging filter with impulse response:

$$h(t) = \frac{1}{T}\Pi_T(t - T/2)$$



- Express the statistical autocorrelation of the input signal $R_{xx}(t, \tau)$. Is X(t) WSS?
- Determine the statistical mean of y(t).
- Show that $R_{yy}(\tau) = \frac{\sigma^2}{T}\Lambda_T(\tau) + A^2$and deduce σ_y and μ_y .
- Take A=2 and plot X(t) and Y(t).

6. We consider the diagram opposite where x(t)=s(t)+b(t) with $s(t) = \Pi_1(t)$ and b(t) is a centered white Gaussian noise with variance 1



- Determine the probability density of x(t) . Is x(t) SSL?

We want to maximize the signal-to-noise ratio (SNR)

- Determine and plot h(t) for k=2 and T_0=2. Then calculate the SNR.
- Which probability law follows y(t) (justify).
- Give a concrete example of the use of this type of filter
- If s(t) is deterministic and unknown, what filter do we use?

7 . We consider the transmission of two symbols:

$s_0(t)=A \quad t\in[0,T]$, $s_1(t)=-A \quad t\in[0,T]$

through a Gaussian additive white noise channel. The signal received is written: x(t)= $s_i(t)$+b(t) where b(t) is a centered white noise of DSP σ_b^2

- Determine the impulse response of the matched filter h(t) corresponding to $s_0(t)$ such that $\int h(t)^2 dt=1$
- Same question for $s_1(t)$. Then, give the signal-to-noise ratio, in each case.

We note $y_{si}(t)$, the output of the filter corresponding to $s_i(t)$,

- Show that for b(t) it is written: $y_b(T)=\frac{1}{\sqrt{T}}\int_0^T b(t)dt$
- Determine the mean and variance of $y_b(T)$
- Assuming that the white noise is Gaussian, determine the probability distribution of the random variable Y= $y_{si}(t)+y_b(t)$

8. Consider a process of the form X(n) = θ + W(n) where W(n) is a Gaussian process with mean 0 and variance 1 such that W(n) and W(j) are independent if n ≠j. We assume that θ follows a law N $(0,\sigma^2)$ independent of W(n) n ∈ Z.
- Characterize the Wiener filter allowing θ to be estimated from X_n, X_{n-1}, ..., X_{n-N+1} .
- Calculate the filter coefficients.

9 . It is desired to de-noise a speech signal z(n) corrupted by additive noise b(n) independent of the sound signal and this, by Wiener filtering. We assume that some autocorrelation values are known for the two signals such that:

$R_{ZZ}[0]$ = 1.5 ; $R_{ZZ}[1]$ = 0.5 ; $R_{ZZ}[2]$ = 0.25 ; $R_{ZZ}[3]$ = 0.125; $R_{ZZ}[4]$ = 0.0625
$R_{bb}[0]$ = 1 ; $R_{bb}[1]$ = 0.25 ; $R_{bb}[2]$ = 0.0625 ; $R_{bb}[3]$ = 0.015625

-Why can't we use a matched filter or an averaging filter?
-Give the Wiener-Hopf equations to estimate z(n).
-Determine the Wiener filter of order 2 allowing to find the useful signal $\hat{z}(n)$then express H(z).

10. Consider a noise estimation problem b(n).
The observed signal is $x(n)=s(n)+b(n)-b(n-1)$.
It is assumed that the statistical correlation of the signal s(n) is $R_{ss}(n)=0.8^{|n|}$and that it is decorrelated from the noise, the autocorrelation of which is $R_{bb}(n)=0.8\,\delta(n)$.

- Determine the statistical means of s*(n)* and *b(n).*
- When is the Wiener filter used?
- Give the Wiener-Hopf equations for estimating *b(n)*
- Determine the Wiener filter of order 2 making it possible to find the useful signal $\hat{b}(n)$.
- Express $\hat{b}(n)$

Solutions

1. See test 1 2016/2017

2. $\mu_z(t)=0 \quad \sigma_z^2(t)=\sigma^2 \qquad R_z(\tau)=\sigma^2\cos(2\pi f_0\tau)$ SSL

3. $R_{xx}(t,\tau)=A^2+\sigma_b^2\delta(\tau)=fct(\tau)$ $\mu_x(t)=A \Rightarrow$SSL $H(f)=sinc(fT)e^{-\pi jfT}$ $\mu_y(t)=A$ averaging filter
4. $H(f)=1/(1+2\pi jfRC)$, $S_y(f)=\sigma_x^2/(1+4(\pi fRC)^2$ $R_y(\tau)=\sigma_x^2/2RC\, e^{-|\tau|/RC}$, $P=R_y(0)=\sigma_x^2/2RC$, interro1 16/17
5. $Rx(0)=(b_0^2+b_1^2+b_2^2+b_3^2)$. $\sigma^2 Rx(1)=(b_0b_1+b_1b_2+b_2b_3)$. $\sigma^2 Rx(2)=(b_0b_2+b_1b_3)$. $\sigma^2 Rx(3)=(b_0b_3)$. $\sigma^2 Rx(k\geq 4)=0$
$H(z)=(z^2-1)/z^2$ $Rxy(0)=2.2$ $Rxy(1)=0.4$ $Rxy(2)=-0.2$ $Rxy(3)=0.1$ $Rxy(-1)=-0.5$ $Rxy(-2)=-2$
6 . x(t) Gaussian with mean s(t) and variance 1. x (t) non stationary $S_{b'}(f)=|H(f)|^2 S_{s'}(f)=|H(f)^2|sinc^2(f)$
$h(t)=2\prod_1(2-t)$ SNR=1 y(t) Gaussian Radar or Sonar Low pass (average)

7 . Question1 14/15 **8** . $R_{xx}(0)=1+\sigma^2$ $R_{xx}(k>0)=\sigma^2$ $R_{\theta x}(k)=\sigma^2$ $b_i=\sigma^2/(N\sigma^2+1)$ **9** . $\mu z=0$, $\mu b=0$, random (speech) signal,
10 . $\mu s=0$, $\mu b=0$, Sb(f)=0.8. When useful signal and noise occupy the same frequency range. $b_0=0.8/2.6$ and $b_1=0$.

Additional exercises

1. Consider the random signal defined by: $x(t)=A_1e^{j2\pi f_1t}+A_2e^{j2\pi f_2t}$ where A_1 and A_2 are two Gaussian vas, uncorrelated, centered and of variance σ^2

- Give the joint ddp $f_{A1,A2}(A1,A2)$
- Determine the statistical mean of the signal x(t) as well as its autocorrelation function and the DSP
- Determine the ddp of x(t)
- Is the process x(t) stationary at the 2nd order ?

2 . The real SSL random signal x(t) has an autocorrelation function of the form $R_x(\tau)=\sigma_x^2e^{-\beta|\tau|}$. Another signal is related to x(t) by the following deterministic equation: y(t) = ax(t) + b, where a and b are given constants.

- What is the autocorrelation function of y(t)? Deduce μ_y and σ_y^2
- What is the cross-correlation and covariance function of x(t) and y(t)?
- Calculate the correlation coefficient. Was this result predictable, why?

3. Which of the following statistic autocorrelation functions can be those of a real random process?

$Rx1(\tau)=\Lambda_2(\tau)-2$	$Rx2(\tau)=-\Lambda_2(\tau)+2$	$Rx3(\tau)=e^{-2|\tau|}$	$Rx4(\tau)=e^{-2\tau}U(\tau)$	$Rx5(\tau)=\delta(\tau-1)+\delta(\tau+$

4 . We consider the signal $x(n)=g(n)\cos(2\pi k_0n/N+\varphi)$ defined for $n\in[0,N-1]$, and where g(n) is a random function SSL, independent of φuniform va ente 0 and 2π.

- Calculate the autocorrelation of x(n)
- Deduce into its power spectral density, as a function of the PSD of g, $S_g(f)$.

5 . Consider the random signal x(t) SSL whose DSP is given by $Sx(f)=\sigma^2B\prod_B(f)$

- Determine the statistical autocorrelation of x(t)
- Deduce the mean and the statistical variance of x(t).

This signal is transmitted through a SLIT whose transfer function $H(f)=\prod_A(f)$ with A <B

- Is the output signal random? (Justify) SSL? (Justify)
- Determine Sy(f) and deduce $Ry(\tau)$ then the statistical moments of order 1 of y(t).

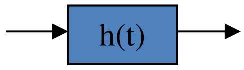
6 . We have a received signal which is the noisy, delayed and attenuated version of a signal of interest s (n). The noise b (n) is assumed Gaussian white of variance σ^2. The problem is to determine the amplitude A and the delay n_0 in the received signal $x(n)=As(n-n_0)+b(n)$. Knowing that the signal-to-noise ratio is maximum at the output of the response matched filter h (n) = s (-n) .

- Verify that the output y (n) of this filter is expressed as the sum of two correlation functions.
- Calculate the variance $\sigma_{b'}{}^2$ of the output noise.
- We take for s (n) a rectangular pulse of width L , then plot an example of received signal and matched filter output.

7. Let $x(t)$ =t with $0 \leq t \leq T$

This signal is used to determine the distance of an object. Knowing that the signal received y(t) by the receiver is delayed by T' and noisy by white Gaussian noise of power spectral density σ^2 :

1. Plot approximately y(t).
2. Determine the general expression for the impulse response of the filter *h(t)* (whose energy is 1) allowing the signal-to-noise ratio to be maximized.
3. Plot h(t) as a function of $T_0 = 2T$ and $k = \sigma^2$, then z(t) the filter output (take T'=10).
4. Give the signal to noise ratio after filtering.
5. Why is this filtering said, in general, optimal and, in particular, Matched?



8 . We consider a process of the form X(n) = b(n) + αb(n-1) + W(n) where W(n) is a Gaussian process with mean 0 and variance σ^2 such that W(n) and W(j) are independent if n ≠j. We assume that b(n) is a uniform random variable with values in {−1,1}, independent of(W(n))n ∈ Z and likewise, b(n) and b(j) are independent if n ≠j (P(b(n) = 1) = P(b(n) = −1) =1/2.

- Build a 3rd order filter that estimates b(n) .

9 . We consider a problem of estimation of a noisy signal s(n).

The observed signal is $x(n)=\alpha s(n)+b(n)$.

We assume that s(n) and b(n) are SSL and decorrelated and that the noise has a PSD $S_{bb}(f)=0.25$.

- Remind the application conditions of Wiener filtering
- Determine the Wiener filter of order 2 making it possible to find the useful signal $\hat{s}(n)$. We will assume that $R_{ss}(k)=2*0.5^{|k|}$
- Express H(z) then $\hat{s}(n)$

10. We consider a problem of estimation of a signal s(n) noisy and having undergone an echo.

The observed signal is $x(n)=s(n)+0.5s(n-1)+b(n)$.

We assume that the autocorrelation of the useful signal is known and that it has the expression $0.5^{|k|}$ and it is assumed that the useful signal is decorrelated from the noise whose autocorrelation is $R_{bb}(k)=0.25^{|k|}$.

1. Determine the statistical means of x *(n)* and *b(n)*
2. Give the Wiener-Hopf equations for estimating *s(n)*
3. Determine the Wiener filter of order 2 making it possible to find the useful signal $\hat{s}(n)$.
4. Express $\hat{s}(n)$and comment

11. We consider the discrete-time linear filter defined by $y(n)=x(n)+b_1x(n-1)+b_2x(n-2)$.

where X(n) and Y (n) respectively designate the real random input and output processes of the filter where b1 and b2 are 2 real coefficients. We assume that x(n) is a sequence of centered ,independent random variables with variance σ^2**.**

- Give the expression of Rx(k) and Sx(f). Give the expression of Ry(k) and plot it for b_1=1 and b_2=-1.
- Knowing the DSP of the signal y(n), what is the basis for the choice of the model?

Solutions

1. See test 1 2014/2015 **2.** $R_y(\tau)=\sigma_x^2 e^{-\beta|\tau|}+b^2$ $S_y(f)=2\beta\sigma_x^2/(\beta^2+4(\pi f)^2)+b^2\delta(f)$ $R_{xy}(\tau)=a R_x(\tau)$

3. See question 1 2015/2016 **4.** $Rx(m)=Rg(m)\cos(2\pi k_0 m/N)/2$ $Sx(f)=[Sg(fk_0)+Sg(f+k_0)]/4$

5 . $R_x(\tau)=\sigma^2 B^2 \mathrm{sinc}(B\tau)$, $\mu_x=0$, $\sigma_x^2=\sigma^2 B^2$, x(t) SSL+h(t) SLIT $\Rightarrow$y(t) SSL, $S_y(f)=\sigma^2 B \prod_A(f)$, $R_x(\tau)=\sigma^2$ $AB\mathrm{sinc}(A\tau)$

6 .$y(n)=AR_s(nn_0)+R_{bs}(n)$ $\sigma_{b'}^2=R_{b'}(0)$ $S_{b'}(f)=\sigma_b^2|H(f)|^2 \Rightarrow R_{b'}(0)=\sigma_{b2}-\int_{-1/2}^{1/2}|H()f|^2 df$

7. Review 15/16 8 **.** $R_{xx}(1{:}3)=(1+\alpha^2+\sigma^2, \alpha, 0)$ $R_{bx}(1{:}3)=(1, 0, 0)$

9. Signals: known useful and stationary and conjointly stationary observation
$R_{xx}(k)=\alpha^2 R_{ss}(k)+R_{bb}(k)$ $R_{sx}(k)=\alpha R_{ss}(k)$

11. $R_x(k)=\sigma_x^2\,\delta(k)$ $S_x(f)=\sigma_x^2$ MA of order 2 $R_y(0)=1+b_1^2+b_2^2$ $R_y(1)=b_1(1+b_2)$ $R_y(2)=b_2$ $R_y(k)=0$ for $k\geq 0$

Exercises Set n° 3: AR modeling and adaptive filtering

1. Consider a random signal y(n) SSL and ergodic whose temporal autocorrelation $\overline{R}_y(k) = \alpha^{|k|}$ with 0< α<1

- Identify the appropriate linear model (AR or MA) for y(n).
- Assuming that the system h(n) is purely recursive filter, give the schematic of the model by defining the input, the system, and the output.
- Recall the necessary assumptions related to the use of this model.
- We consider that the model is of order 1, determine its parameters.

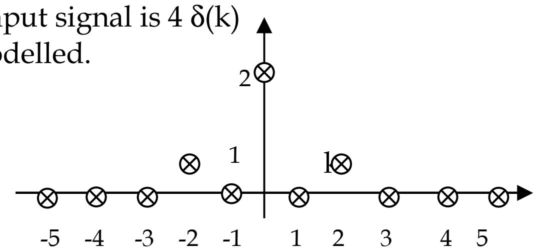
2. Consider an AR process defined by: $y(n)=-a_1\,y(n-1) - a_2\,y(n-2) + x(n)$ where x(n) bb decorrelated with variance 1

- Calculate $\mu_y(n)$ then without calculation, explain why y(n) is SSL.
- Show that for k>0, $Ry(k)=-a_1\,Ry(k-1)-a_2\,Ry(k-2)$
- Determine a_1 and a_2

3. We want to model the signal y(n) whose statistical autocorrelation is given in the figure opposite. Ry(k). It is assumed that the statistical autocorrelation of the input signal is 4 δ(k)

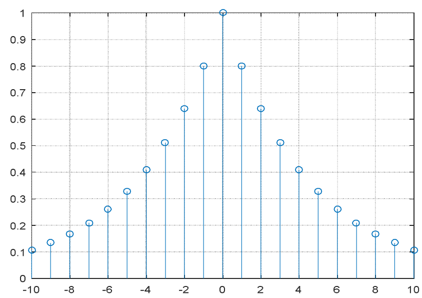
- Determine the first-order statistical moments of the signal to be modelled.
- Is this an AR or MA model? Justify
- Determine the order of the model

Assuming that 2 coefficients are equal, from the values of Ry(k), deduce that one of the coefficients is zero.
then determine the coefficients of this model then give the difference equation linking y(n) and x(n)



4 . We want to model the signal y(n) SSL whose statistical correlation has only been plotted from -10 to 10. It is given in the figure opposite.

- Is this an AR or MA model? Justify
- Give the statistical properties of the input signal.
- We assume that the model is of order 1, determine its parameters.
- Explain the concept of shaping filter and its usefulness in telephony



5. We consider a target signal $y[n]$, a noisy observation $x[n] = y[n] + b[n]$. where $b[n]$is additive noise.
For all exercises, use these few values (length 5) for both signals :

- Target signal $y = [1, 2, 0, -1, 3]$
- Noisy observation: $x = [1.2, 1.9, 0.1, -0.8, 2.7]$
- FIR filter length: N = 3 (Second order filter)

1. Wiener Filter by Matrix Inversion

For a length-3 filter: $h = [h_0\ h_1\ h_2]^T$ or $h = [b_0\ b_1\ b_2]^T$
The optimal Wiener solution: $h_{\text{opt}} = R_{xx}^{-1}R_{yx}$

- Compute the autocorrelation matrix : $R_{xx} = E[x(n)x^T(n)]$ → Empirique $R_{xx} = \frac{1}{N}X^TX$
- Compute the cross-correlation vector $R_{yx} = E[y(n)x(n)]$ → Empirique $R_{yx} = \frac{1}{N}X^Ty.$
- Solve $h = R_{xx}^{-1}R_{yx}$
- Filter $y[n]$with your h and compute the estimation error $e[n] = x[n] - \hat{x}[n]$

2. Steepest Descent Method

Steepest descent update: $h_{k+1} = h_k + \mu(R_{yx} - R_{xx}h_k)$

- Use the $R_{xx}$and R_{yx} computed in Exercise 1.
- Initialize $h_0 = [0\ 0\ 0]^T$.
- Compute the largest eigenvalue $\lambda_{\max}$of R_{xx}(use a calculator or Python).
- Choose $\mu = \frac{1}{\lambda_{\max}}$
- Perform **10 iterations** of steepest descent manually or in Python.
- Plot or tabulate the Mean Square Error after each iteration: $MSE(k) = \frac{1}{5}\sum(x[n] - \hat{x}_k[n])^2$

3. LMS (Instantaneous Gradient)

LMS update: $e[n] = y[n] - h^T(n)x(n)$ → $h(n+1) = h(n) + \mu\, e[n]\, x(n)$

- Build the input vector $x(n) = [x[n], x[n-1], x[n-2]]^T$ using zero-padding for negative indices.
- Initialize $h(0) = [0\ 0\ 0]^T$
- Choose a stable step size $\mu = 0.05$
- Run LMS for samples $n = 0 \dots 4$. After all updates, write the final value of the filter h.
- Compare the LMS estimate with
  - the steepest-descent filter
  - the Wiener closed-form solution.

4. Comparison and Discussion

- Compare the three filters numerically.
- Explain:
  - Which method converges fastest?
  - Which method is computationally cheapest?
  - Which method is most sensitive to step size?
- Modify the noise level in y(e.g., $y = x + 0.5b$) and repeat LMS.
- Comment on performance changes.

6. Explain and jutify the following assessment put in the table below

Method	Convergence Speed	Simplicity / Cost	Robustness	Practical Example	Main Limitation
Wiener	Immediate	Moderate	Good	Noise removal in pre-recorded audio or ECG, image restoration	Offline only, requires exact correlation knowledge
Steepest Descent (SD)	Slow	Simple	Moderate	Channel equalization in telecom, adaptive filtering of static audio noise	Slow convergence, sensitive to step size
LMS	Moderate	Very simple, low cost	Good	ANC (microphone), real-time channel equalization, adaptive tracking of portable biological signals	Irregular convergence, step size must be tuned
RLS	Very fast	More complex, higher cost	Excellent	High-speed channel equalization, radar/sonar, precise adaptive tracking of industrial or biological signals	Computationally heavy, requires recursive matrix inversion

Solutions

1. AR model ($R_y(k)\neq 0$) x(n) input bb, y(n) random signal to be modeled h(n) shaping filter (math model) input bb + ergodism $a_1 = -\alpha \sigma_x{}^2 = 1-\alpha^2$

2. $\mu_y(n)=0$ input bb SSL ⇒output SSL Interro1 14/15

3. See exam 17/18 **4.** Exam 16/17

5.

5.1. Wiener solution Autocorrelation of the input (noisy observation) $R_{xx}$is 3 × 3:

$$R_{xx} = \begin{bmatrix} r_{xx}[0] & r_{xx}[1] & r_{xx}[2] \\ r_{xx}[1] & r_{xx}[0] & r_{xx}[1] \\ r_{xx}[2] & r_{xx}[1] & r_{xx}[0] \end{bmatrix}$$ Where $r_{xx}[k]$is the autocorrelation of $x[n]$: $r_{xx}[k] = \frac{1}{N}\sum_{n=k}^{N_y-1} x[n]\cdot x[n-k]$

Hence: $R_{xx} = \begin{bmatrix} 2.6 & 0.046 & -0.226 \\ 0.046 & 2.6 & 0.046 \\ -0.226 & 0.046 & 2.6 \end{bmatrix}$

Cross-correlation vector R_{yx} → $R_{yx} = \begin{bmatrix} r_{yx}[0] \\ r_{yx}[1] \\ r_{yx}[2] \end{bmatrix}$, $r_{yx}[k] = \sum_{n=k}^{4} y[n]x[n-k]$ → R_{yx} =[2.78, -0.02, -0.32]^T

Wiener Filter Solution: h = [1.0673, -0.0261, -0.0299]^T

5.2. Steepest-descent (deterministic gradient) — iterative approach

Algorithm: $h_{k+1} = h_k + \mu\,(R_{yx} - R_{xx}h_k)$

Choose $\mu = 1/\lambda_{\max}$(safe step);

Eigen values of R_{xx} ≈ [2.826, 2.6, 2.374] in this case $\lambda_{\max}$ ≈ 3.1168, so $\mu < 0.707$ → μ = 0.1

Iteration 1:

$R_{xx}h_0 = [0, 0, 0]^T$ $\quad R_{yx} - R_{xx}h_0 = [2.78, -0.02, -0.32]^T$

$h_1 = [0, 0, 0]^T + \mathbf{0.1} \times [2.78, -0.02, -0.32]^T = [0.278, -0.002, -0.032]^T$

Iteration 2:

$R_{xx}h_1$ = [2.6×0.278 + 0.046×(-0.002) + (-0.226)×(-0.032), 0.046×0.278 + 2.6×(-0.002) + 0.046×(-0.032), -0.226×0.278 + 0.046×(-0.002) + 2.6×(-0.032)]

$R_{xx}h_1$ = [0.7228 - 0.000092 + 0.007232, 0.012788 - 0.0052 - 0.001472, -0.062828 - 0.000092 - 0.0832]

$R_{xx}h_1$ = [0.72994, 0.006116, -0.14612]

$R_{yx} - R_{xx}h_1$= [2.78 - 0.72994, -0.02 - 0.006116, -0.32 - (-0.14612)] = [2.05006, -0.026116, -0.17388]

h_2 = [0.278, -0.002, -0.032]^T + **0.1**×[2.05006, -0.026116, -0.17388]^T = [0.278 + 0.20501, -0.002 - 0.002612, -0.032 - 0.017388]

= [0.48301, -0.004612, -0.049388]^T

Iteration 3:

$R_{xx}h_2$ = [2.6×0.48301 + 0.046×(-0.004612) + (-0.226)×(-0.049388), 0.046×0.48301 + 2.6×(-0.004612) + 0.046×(-0.049388), -0.226×0.48301 + 0.046×(-0.004612) + 2.6×(-0.049388)]

= [1.25583 - 0.000212 + 0.011162, 0.022218 - 0.011991 - 0.002272, -0.10916 - 0.000212 - 0.12841]

= [1.26678, 0.007955, -0.23778]

$R_{yx} - R_{xx}h_2$= [2.78 - 1.26678, -0.02 - 0.007955, -0.32 - (-0.23778)] = [1.51322, -0.027955, -0.08222]

h_3 = [0.48301, -0.004612, -0.049388]^T + **0.1**×[1.51322, -0.027955, -0.08222]^T

= [0.48301 + 0.151322, -0.004612 - 0.002796, -0.049388 - 0.008222] = [0.634332, -0.007408, -0.05761]^T

After several iterations, it converges to the Wiener solution.

Iteration	h[0]	h[1]	h[2]	MSE
1	0.2780	-0.0020	-0.0320	**0.8421**
2	0.4830	-0.0046	-0.0494	**0.5923**
3	0.6343	-0.0074	-0.0576	**0.4318**
4	0.7461	-0.0101	-0.0603	**0.2847**
5	0.8288	-0.0127	-0.0597	**0.2361**
6	0.8937	-0.0147	-0.0583	**0.1983**
7	0.9438	-0.0161	-0.0563	**0.1698**
8	0.9819	-0.0171	-0.0541	**0.1487**
9	1.0105	-0.0178	-0.0518	**0.1333**
10	1.0315	-0.0183	-0.0495	**0.1222**

- ➔ Wiener Filter Solution: h = $[1.0673, -0.0261, -0.0299]^T$
- ➔ Steepest-descent converges toward the Wiener solution; rate depends on μ and eigenvalue spread.

5.3. LMS (instantaneous gradient) — one pass example

LMS update (we estimate y from x):

LMS: $h_{k+1} = h_k + \mu e_k x_k$, where $e_k = y_k - h_k^T x_k$

$$e[n] = y[n] - h^T(n)\,\mathbf{x}(n) \quad \rightarrow h(n+1) = h(n) + \mu\, e[n]\,\mathbf{x}(n)$$

Take $\mu = 0.01$, Initial guess: $h_0 = [0, 0, 0]^T$

Iteration 1 (n=0):

$x_0 = [1.2, 0, 0]^T$ (assuming zero initial conditions) $y_0 = 1$

→ $e_0 = 1 - [0, 0, 0]^T[1.2, 0, 0] = 1$ → $h_1 = [0, 0, 0]^T + 0.01\times1\times[1.2, 0, 0]^T = [0.012, 0, 0]^T$

Iteration 2 (n=1):

$x_1 = [1.9, 1.2, 0]^T$ $y_1 = 2$

→ $e_1 = 2 - [0.012, 0, 0]^T[1.9, 1.2, 0] = 2 - 0.0228 = 1.9772$ → $h_2 = [0.012, 0, 0]^T + 0.01\times1.9772\times[1.9, 1.2, 0]^T = [0.04957, 0.02373, 0]^T$

Iteration 3 (n=2):

$x_2 = [0.1, 1.9, 1.2]^T$ $y_2 = 0$

→ $e_2 = 0 - [0.04957, 0.02373, 0]^T[0.1, 1.9, 1.2] = 0 - (0.004957 + 0.045087 + 0) = -0.050044$

→ $h_3 = [0.04957, 0.02373, 0]^T + 0.01\times(-0.050044)\times[0.1, 1.9, 1.2]^T = [0.04952, 0.02278, -0.0006005]^T$

Continue this process for all data points and multiple epochs until convergence.

Take $\mu = 0.1$

Epoch	Iterations	h[0]	h[1]	h[2]	MSE
1	5	0.3851	0.0241	-0.0343	1.4278
2	10	0.6723	0.0039	-0.0508	0.5236
3	15	0.8486	-0.0089	-0.0513	0.2045
4	20	0.9497	-0.0164	-0.0457	0.0857
5	25	1.0052	-0.0203	-0.0388	0.0398
6	30	1.0351	-0.0224	-0.0332	0.0218
7	35	1.0507	-0.0236	-0.0297	0.0144
8	40	1.0589	-0.0242	-0.0278	0.0110
9	45	1.0633	-0.0246	-0.0268	0.0093
10	50	1.0657	-0.0248	-0.0263	0.0084
15	75	1.0672	-0.0251	-0.0255	0.0068
20	100	1.0673	-0.0252	-0.0253	0.0065

5.4. Summary of Results:

- Steepest Descent: Converges to Wiener solution
- LMS: Converges to Wiener solution with proper step size and sufficient iterations
- The Wiener filter gives the optimal solution in one step, while steepest descent and LMS are iterative methods that converge to the same solution.
- Choice of μ matters: too large → instability; too small → very slow convergence.

Additional exercises

1. We consider a stationary random signal x(n) and we assume its autocorrelation coefficients are known: $R(0) = 3\sigma^2$, $R(1) = 2\sigma^2$, $R(2) = \sigma^2$, $R(3) = 0$

- We are looking for the MA filter of order 3 of this signal. Identify the filter settings.
- If the signal x(n) was obtained by filtering a white Gaussian noise of variance σ^2 by a finite impulse response filter with transfer function $H(z) = 1 + az^{-1} + bz^{-2}$, deduce the values of a and b.

2. Consider a shaping filter whose difference equation is y(n)=0.5 (x(n)+x(n-1)+x(n-2)+x(n-3))

- Explain the notion of shaping filter then identify this linear model AR or MA
- Give the mean, the autocorrelation and the PSD of its input *x(n)*.
- Calculate and plot $R_{yy}(k)$ then determine $S_{yy}(f)$

3 . The signals x(n) and y(n) were obtained by filtering, by means of a finite impulse response filter, a centered Gaussian white noise b(n) of variance σ^2.

x (n) = 2.b(n) + 0.5.b(n -1) -0.2.b(n - 2) + 0.1.b(n - 3)   y(n) = b(n) - b(n-2)

- Identify the 2 models then for each, calculate and plot the autocorrelation coefficients
- are x(n) and y(n) Gaussian (justify)
- Calculate the intercorrelations Rxy(k), and Ryx(k) and comment
- Give some applications of the AR, MA, ARMA models

4 . Consider a shaper filter whose difference equation is y(n)= - αy(n-1) - βy(n-2) + x(n)

- Identify the order of the AR linear model.
- Determine the parameters of the models, it is assumed that $R_{yy}(k)=2*0.5^{|k|}$

5 . Consider a shaper filter whose difference equation is y(n)= αy(n-1) + x(n)

- Identify the order of the AR linear model.
- Determine the mean of y(n) and show that $R_{yy}(k)= \alpha^k R_{yy}(0)$, deduce a condition on α.
- Show that $R_{yy}(0)=R_{xx}(0)/(1-\alpha^2)$
- Plot $R_{yy}(k)$ (Take $R_{xx}(0)=\sigma^2=1$).
- Name 2 concrete applications of AR models.

6 . Consider a shaping filter whose difference equation is y(n)= 0.25 y(n-1) - 0.25 y(n-2) + x(n)

- Identify this AR or MA linear model (Justify)
- Determine the mean of y(n) and give the expression for $R_{yy}(k)$.
- Compute and plot $R_{yy}(k)$ (Take $R_{xx}(0)=\sigma^2=1$).

Solutions

1. $b_0=1$ $b_1=1$ and $b_2=1$ and by identification, we find a=1 and b=1

2. Question 2 15/16 **3.** See 15/16   **4.** Ratt 14/15

Exercises set n°4: Time-Frequency and Time-Scale Transforms

1. Determine the TFCT of the following signal (h is a gate window of width T):

$$x(t) = \begin{cases} e^{2\pi j f_1 t} & pour\ t < t_0 \\ e^{2\pi j f_2 t} & pour\ t > t_0 \end{cases}$$

2. Calculate the Wigner-Ville transform of the signal$x(t) = A_1 e^{2\pi j f_1 t} + A_2 e^{2\pi j f_2 t}$

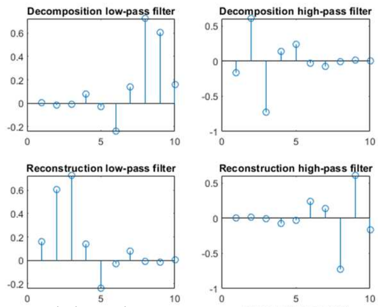
3. Consider the following wavelets:

- Haar's function $\psi(t) = \begin{cases} 1 & si & 0 \leq t \leq \frac{1}{2} \\ -1 & si & \frac{1}{2} \leq t \leq 1 \\ 0 & & ailleurs \end{cases}$
- The Mexican hat wavelet $\psi(t) = \alpha(1 - t^2)e^{-\frac{t^2}{2}}$

Calculate their TF and plot the

4. Given the following low-pass decomposition filter: h_0 ={-0.1294, 0.2241, 0.8365, 0.4830}, determine the high-pass decomposition filter and the reconstruction filters

5. Let the decomposition and reconstruction filters be following, check the properties linking the 4 filters

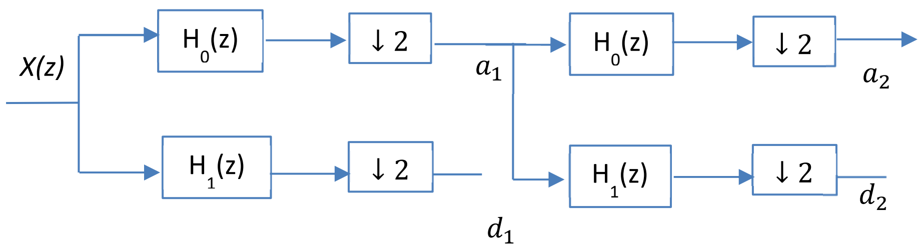


The four filters for db5

6. Let the following signal x ={2, 4, 8, 12, 14, 0, 2, 1} give and trace its decomposition into wavelets if the scale function is h_0=[0.5, 0.5]

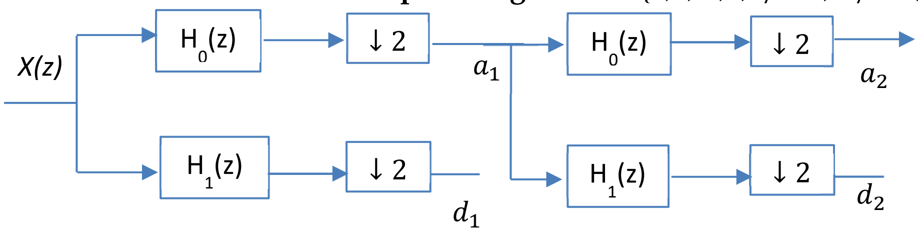
7. Consider the following signal:$f = \{4, 6, 10, 12, 8, 6, 5, 5\}$
- Give and plot its decomposition at level 1 into wavelets using the Haar wavelet.
- Reconstruct the signal from level 1
- Calculate its energy and then that after decomposition.
- Check that it is conserved and that 98% of this energy is in the approximated signal.
- Give the decomposition at levels 2 then 3.

8. A signal x(n) is decomposed by the Haar wavelet $h_0 = \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$according to the diagram below . The first 8 values of the signal x(n) are ={4, 4, 6, 8, 8, 10, 9, 11}



- Determine a_1, d_1, a_2, d_2, (**Detailed calculation**) then deduce the decomposed signal.
- Assuming that f_e=8kHz, give the frequency bands occupied by a_1, d_1, a_2, d_2,
- Give the reconstruction scheme and the expression of filters $F_0(z)$ and $F_1(z)$.

9. We have decomposed a signal x(n) by the Haar wavelet $h_0 = \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$according to the diagram below . The first 8 values of the **decomposed signal** are ={4,4,-1,1,2/ $\sqrt{2}$,-1/ $\sqrt{2}$,,0,1/ $\sqrt{2}$}



- Give the reconstruction scheme and the expression of the filters $F_0(z)$ and $F_1(z)$.
- Assuming that f_e=18kHz, give the frequency bands occupied by a_1, d_1, a_2, d_2,
- Identify a_1, d_1, a_2, d_2then **reconstruct** the original signal

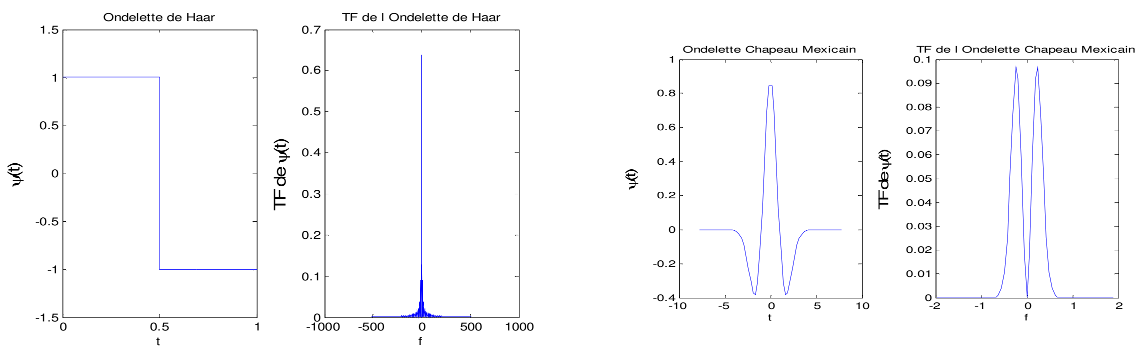
Solutions

1.

$$x(t) = \begin{cases} T sinc\big((f-f_1)T\big)e^{-2\pi j(f-f_1)t} & pour\ t+\frac{T}{2} < t_0 \\ T sinc\big((f-f_2)T\big)e^{-2\pi j(f-f_2)t} & pour\ t-\frac{T}{2} > t_0 \\ \left(t_0 - t + \frac{T}{2}\right) sinc\left(\left(t_0 - t + \frac{T}{2}\right)(f-f_1)T\right) e^{-2\pi j(f-f_1)\left(\frac{t_0+t}{2}-\frac{T}{4}\right)} + & \\ \left(\frac{T}{2} - t_0 + t\right) sinc\left(\left(\frac{T}{2} - t_0 + t\right)(f-f_2)T\right) e^{-2\pi j(f-f_2)\left(\frac{t_0+t}{2}+\frac{T}{4}\right)} & pour\quad -\frac{T}{2} < t - t_0 < \frac{T}{2} \end{cases}$$

2. $WV_x(t,f) = |A_1|^2\delta(f-f_1) + |A_2|^2\delta(f-f_2) + 2A_1A_2\cos(2\pi(f_1-f_2))\delta\left(f - \frac{(f_1+f_2)}{2}\right)$

3. Haar $\hat{\psi}(f) = j\frac{\sin^2\left(\frac{\pi f}{2}\right)}{\frac{\pi f}{2}}e^{-\pi j f}$ Mexican hat$\hat{\psi}(f) = \beta 4\pi^2 f^2 e^{-\frac{f^2}{2}}$



4. h_1 ={-0.4830, 0.8365, -0.2241, - 0.1294}, f_0={0.4830, 0.8365, 0.2241, - 0.1294}, f_1 ={-0.1294, -0.2241, 0.8365, -0.4830}
5.
6.

Level	Approximation	Details
0	2 4 8 12 14 0 2 1	
1	3 10 7 1.5	-1 -2 7 0.5
2	6.5 4.25	-3.5 2.75
3	5.375	1.125

Decomposed signal: {5.375, 1.125, -3.5, 2.75, -1, -2, 7, 0.5}

7. **Decomposition h_0** Filters = $\left[\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\right]$, h_1= , $\left[-\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\right]$ **Reconstruction f_0** Filters = $\left[\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\right]$, f_1=$\left[\frac{1}{\sqrt{2}}\ -\frac{1}{\sqrt{2}}\right]$

Decomposition

f: 4 6 10 12 8 6 5 5 → 5, 11, 7, 5 → $\mathbf{a}^1$: $5\sqrt{2}$, $11\sqrt{2}$, $7\sqrt{2}$, $5\sqrt{2}$.

f: 4 6 10 12 8 6 5 5 → −1, −1, 1, 0 → $\mathbf{d}^1$: $-\sqrt{2}$, $-\sqrt{2}$, $\sqrt{2}$, 0.

$$(4,6,10,12,8,6,5,5) \xrightarrow{\mathbf{H_1}} (5\sqrt{2}, 11\sqrt{2}, 7\sqrt{2}, 5\sqrt{2}\,|\,-\sqrt{2}, -\sqrt{2}, \sqrt{2}, 0)$$

Reconstruction

$\mathbf{a}^1$: $5\sqrt{2}$ $11\sqrt{2}$ $7\sqrt{2}$ $5\sqrt{2}$
$\mathbf{d}^1$: $-\sqrt{2}$ $-\sqrt{2}$ $\sqrt{2}$ 0
f: 4 6 10 12 8 6 5 5.

Energy

$\mathcal{E}_\mathbf{f} = 4^2 + 6^2 + \cdots + 5^2 = 446.$

$\mathcal{E}_{(\mathbf{a}^1\,|\,\mathbf{d}^1)} = 25 \cdot 2 + 121 \cdot 2 + \cdots + 2 + 0 = 446$

$\mathcal{E}_{\mathbf{a}^1} = 25 \cdot 2 + 121 \cdot 2 + 49 \cdot 2 + 25 \cdot 2 = 440$ $\mathcal{E}_{\mathbf{d}^1} = 2 + 2 + 2 + 0 = 6$

$\mathbf{a}^1$: $5\sqrt{2}$ $11\sqrt{2}$ $7\sqrt{2}$ $5\sqrt{2}$ → $\mathbf{a}^2$: 16, 12

$\mathbf{a}^1$: $5\sqrt{2}$ $11\sqrt{2}$ $7\sqrt{2}$ $5\sqrt{2}$ → $\mathbf{d}^2$: −6, 2

$(\mathbf{a}^2\,|\,\mathbf{d}^2\,|\,\mathbf{d}^1) = (16, 12\,|\,-6, 2\,|\,-\sqrt{2}, -\sqrt{2}, \sqrt{2}, 0)$

$(\mathbf{a}^3\,|\,\mathbf{d}^3\,|\,\mathbf{d}^2\,|\,\mathbf{d}^1) = (14\sqrt{2}\,|\,2\sqrt{2}\,|\,-6, 2\,|\,-\sqrt{2}, -\sqrt{2}, \sqrt{2}, 0)$

88% of the energy is contained in a ³ representing yet the 8th ᵒᶠ the signal

8. Review 19/20 **9.** Ratt 19/20

Additional Exercises

1. Wigner-Ville

– Signal gaussien

$$z(t) = \frac{1}{\sqrt{\sigma}} e^{-\pi t^2/\sigma^2} \longrightarrow W_z(t,f) = \sqrt{2}\, e^{-2\pi(t^2/\sigma^2 + \sigma^2 f^2)}$$

– Signal monochromatique d'enveloppe gaussienne

$$z(t) = \frac{1}{\sqrt{\sigma}} e^{-\pi(t-t_0)^2/\sigma^2}\, e^{2i\pi f_0 t} \longrightarrow W_z(t,f) = \sqrt{2}\, e^{-2\pi\alpha(t-t_0)^2/\sigma^2 - 2\pi\sigma^2(f-f_0)^2}$$

– Chirp idéal

$$z(t) = e^{i\pi\beta t^2 + 2i\pi f_0 t} \longrightarrow W_z(t,f) = \delta(f - \beta t - f_0)$$

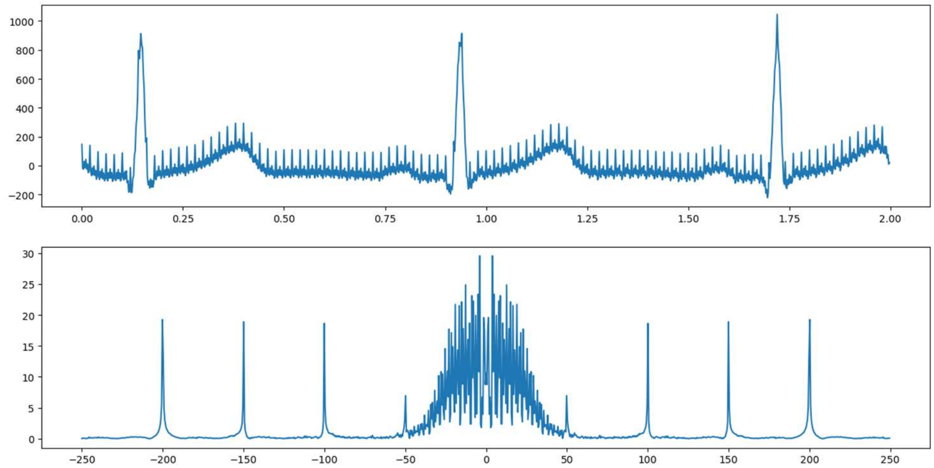
2. Consider the following signal: x(n)={0, 1, -2, 3, -4, 5, -6, 7 }
- Give its wavelet decomposition at level 1 using the Haar wavelet such that the decomposition low-pass filter is $h_0 = \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$
- Give the reconstruction diagram allowing to pass from one level to another.
- Rebuild the signal from level 1.
- Give the decomposed signal at levels 2 then 3.
- Check that the energy is conserved at all levels (1, 2 and 3)
- Propose a way to compress this signal.

3. Consider the following signal: x(n)={ 11, 9, 5, 7, 5, 11, 7, 9 }
- Name a disadvantage of the continuous wavelet.
- Give the discrete wavelet decomposition scheme for 2 levels.
- Give the corresponding diagram for the frequency distribution.
- Give its wavelet decomposition at level 2 using the following filters:

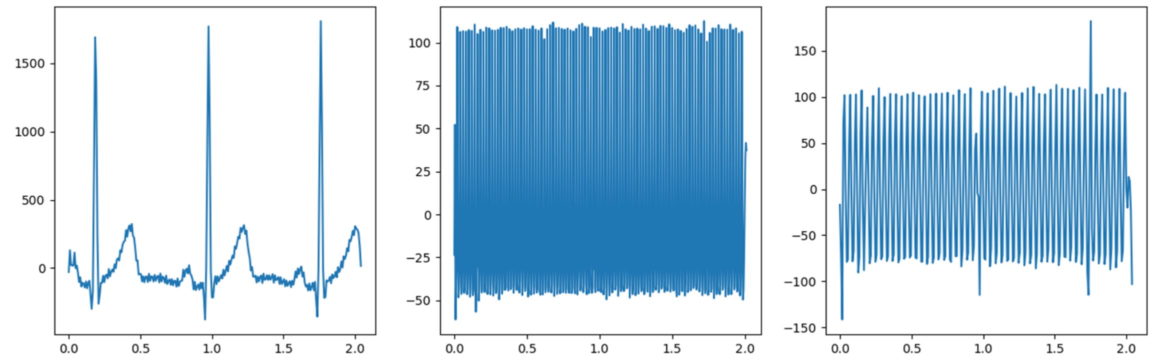
h_0={0.5, 0.5} , h_1={-0.5, 0.5} , f_0=2*{0.5, 0.5} , f_1=2*{0.5, -0.5}

- Is energy conserved?
- Rebuild the signal from level 2.

4. During the transmission of an ecg signal, it was affected by a noise resulting from the combination of 3 sinusoids (see figure below: noisy signal + Modulus of its DFT).



By analyzing the signal by dyadic discrete wavelets, we obtained the following graphs:



- Identify for each of the 3 signals: the signal (approximate or detailed), the level of breakdown
- For each of the 3 signals, sketch the module of the DFT
- Propose a solution to remove the noise (detailed explanation)

Workarounds:
2. review 17/18
3.Ratt 17/18
4 . Review 20/21