# Chapter IV TAS
# Time-Frequency/Scale Analysis

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# I. Introduction

Real-world signals (speech signals and images) are not stationary.

Essential information in the evolution of their characteristics (statistical, frequency, temporal, spatial)

Fourier analysis

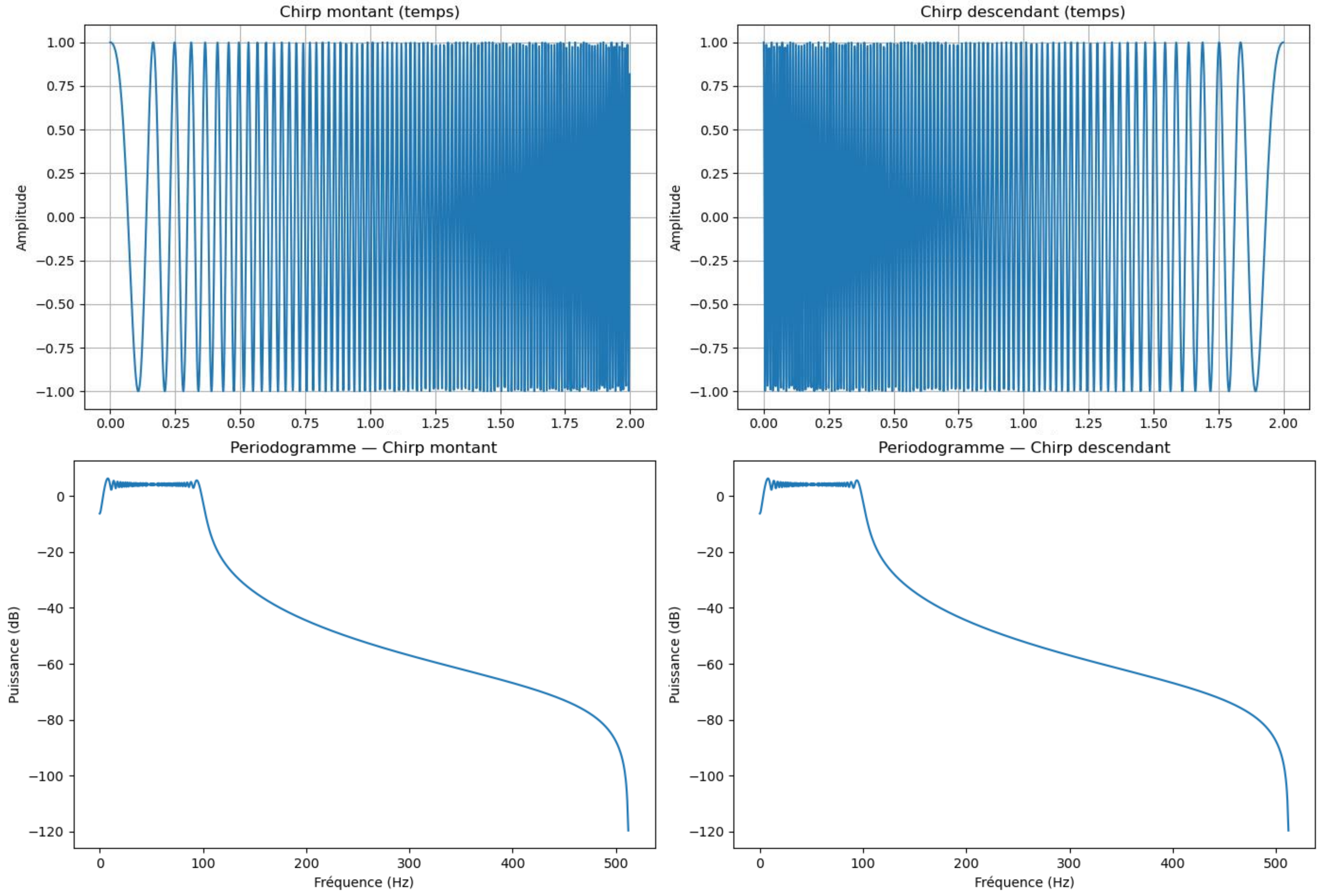
- Global characterization of the signal (integral from -∞ to +∞),
- No temporal or spatial localization

Solution: A transformation that provides information on the frequency content while preserving the location (temporal or spatial) $\Rightarrow$ Time/frequency or space/scale representation

- Windowed Fourier Transform
- Wigner-Ville Transform
- Wavelet Transform.

# I. Introduction

**Example** : TF/Periodogram of a Chirp going from f1 then f2 and vice versa




➢No temporal localization of frequency content)

# II. Short-term Fourier Transform

- Apply TF to small portions $\Rightarrow$ Local spectrum

$$TFCT(t,f)=\int_{-\infty}^{+\infty}x(\tau)h^{*}(\tau-t)e^{-j2\pi f\tau}d\tau$$

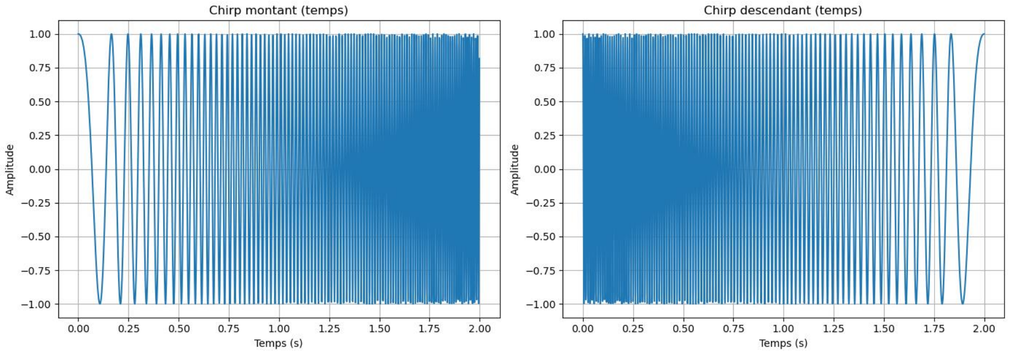
*h(t): weighting window ( Hammng , Blackman ,..) Spectrogram*

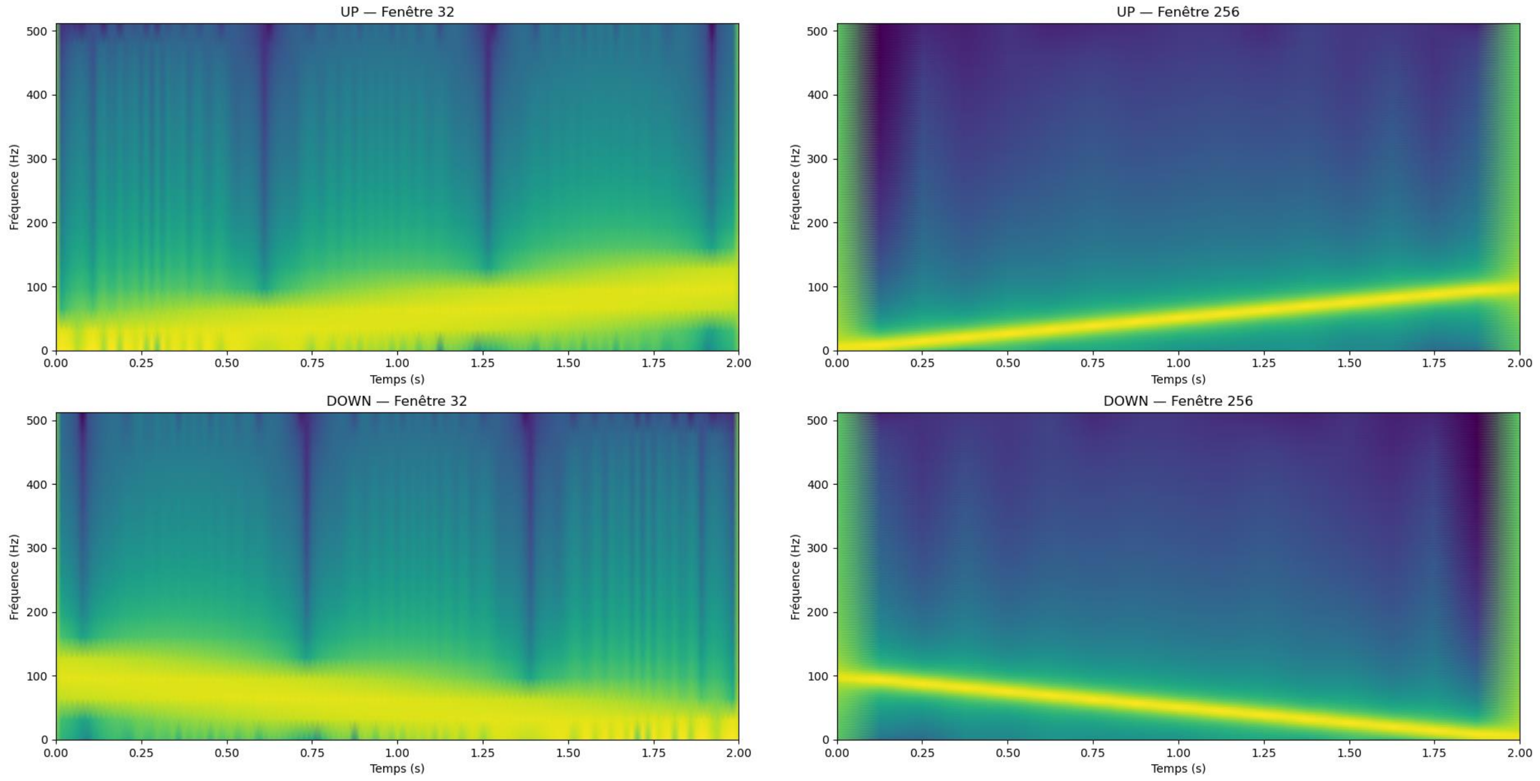
$$S_x(t,f)=\left|\int_{-\infty}^{+\infty}x(\tau)h^{*}(\tau-t)e^{-j2\pi f\tau}d\tau\right|^2$$

- Choosing the right window (main lobe width, side lobe amplitude)
- Overlap rate between windows
- Trade-off Time Resolution - Frequency Resolution (Gaussian Better)

# II. Short-term Fourier Transform

**Example** : **TFCT** of a Chirp going from f1 then f2 and vice versa




- Localization of energy distribution simultaneously in time and frequency
- Time and frequency resolutions evolve inversely to each other
- Single window size choice problem

# III. Wigner-Ville Transform

- Reminder: Spectral density

$S_x(f) = |X(f)|^2 = TF\{R_x(\tau)\}$ with $R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)dt$

- Instant correlation $x(t+\tau/2)x^*(t-\tau/2)$

$\Rightarrow$ Instantaneous spectral density $WV_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-2\pi jf\tau}d\tau$

- Many of the properties of TF retained except *the linearity property* :

$x(t) = \sum_{k=1}^{N} x_k(t) \rightarrow WV_x(t,f) = \sum_{k=1}^{N} WV_{x_k}(t,f) + \sum_{k,m=1(k\neq m)}^{N} WV_{x_k x_m}(t,f)$

- Creation of interference (no real existence)

- Appearance of negative energies $\Rightarrow$ Pseudo-smoothed versions to alleviate these problems.

# III. Wigner-Ville Transform

**Examples**

$WV_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-2\pi jf\tau}d\tau$

-$x(t) = e^{2\pi jf_0t}$ → $WV_x(t,f) = \delta(f-f_0)$

-$y(t) = \cos(2\pi f_0 t) = \frac{e^{2\pi jf_0t}+e^{-2\pi jf_0t}}{2}$

$$WV_y(t,f) = \frac{1}{4}\left(\int_{-\infty}^{\infty}\left[e^{2\pi jf_0(t+\frac{\tau}{2})}+e^{-2\pi jf_0(t+\frac{\tau}{2})}\right]\left[e^{-2\pi jf_0(t-\frac{\tau}{2})}+e^{2\pi jf_0(t-\frac{\tau}{2})}\right]e^{-2\pi jf\tau}d\tau\right)$$

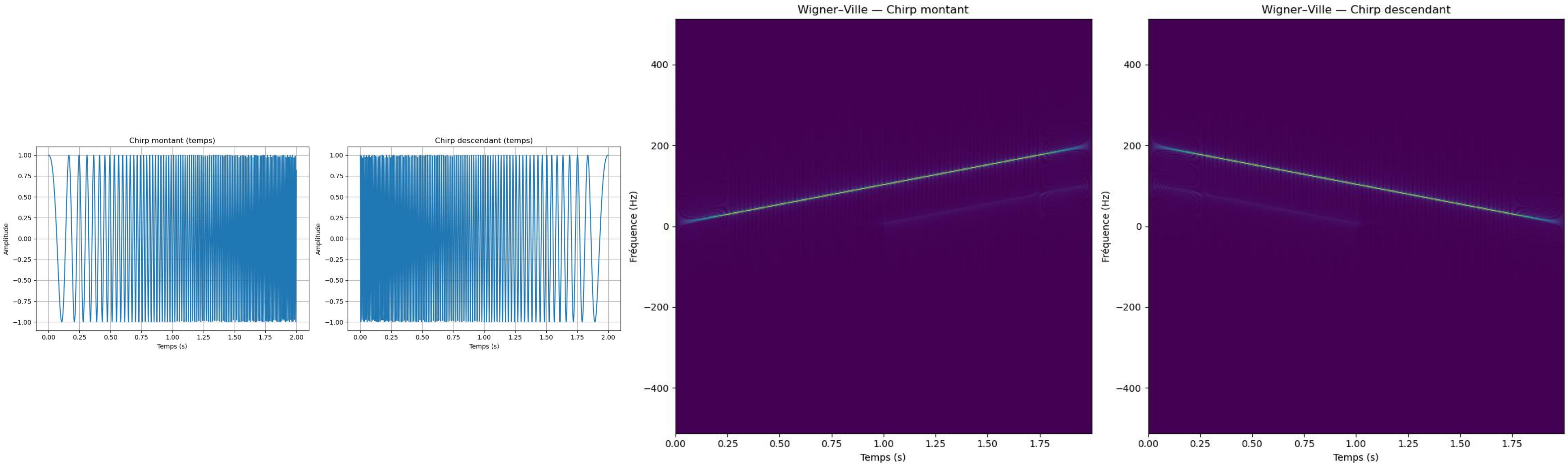
$$WV_y(t,f) = \frac{1}{4}\left(\int_{-\infty}^{\infty} e^{2\pi jf_0\tau}\,e^{-2\pi jf\tau}d\tau + \int_{-\infty}^{\infty} e^{4\pi jf_0t}\,e^{-2\pi jf\tau}d\tau + \int_{-\infty}^{\infty} e^{-4\pi jf_0t}\,e^{-2\pi jf\tau}d\tau + \int_{-\infty}^{\infty} e^{-2\pi jf_0\tau}\,e^{-2\pi jf\tau}d\tau\right)$$

$$WV_y(t,f) = \frac{1}{4}\left(\delta(f-f_0) + e^{4\pi jf_0t}\delta(f) + e^{-4\pi jf_0t}\delta(f) + \delta(f-f_0)\right)$$

→$WV_y(t,f) = \frac{1}{4}\left(\delta(f-f_0)+\delta(f+f_0)\right) + 2\cos(4\pi f_0 t)\delta(f)$

# III. Wigner-Ville Transform

**Example** : **VW** of a Chirp going from f1 then f2 and vice versa




- Good frequency localization despite interference

# IV. Continuous Wavelets

**TFCT**

- Time and frequency resolutions evolve inversely to each other
- Single window size choice problem

**Wigner-Ville**

- Creation of interference (no real existence)
- Appearance of negative energies

**Solution**

Choose a window and a waveform (signal oscillating in a given time window) that could be expanded (for low frequencies) and contracted (high frequencies) at will $\Rightarrow$ Continuous Wavelets

# IV. Continuous Wavelets

Introduced by *Jean Morlet* in 1981 to solve seismic signal problems in oil exploration.

Starting from a window $\psi(t)$ ( called mother function) $\Rightarrow$ Generate a **set** of similar basis functions by dilation (index a) and translation (index *b* ) of a single prototype $\psi_{a,b}(t)$ :

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) \qquad a > 0$$

- ***a > 0*** is a scale parameter **of contraction** ( $a < 1$ ) or **dilation** ( $a > 1$ ) of the window
- ***b*** a **translation** of the window.

By decomposing the signal *x(t)* on this family, we obtain the wavelet coefficients

$$WT_{x,\psi}(a,b) = \int_{-\infty}^{\infty} x(t)\psi_{a,b}^{*}(t)\ dt = \frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} x(t)\psi\left(\frac{t-b}{a}\right)^{*}\ dt$$

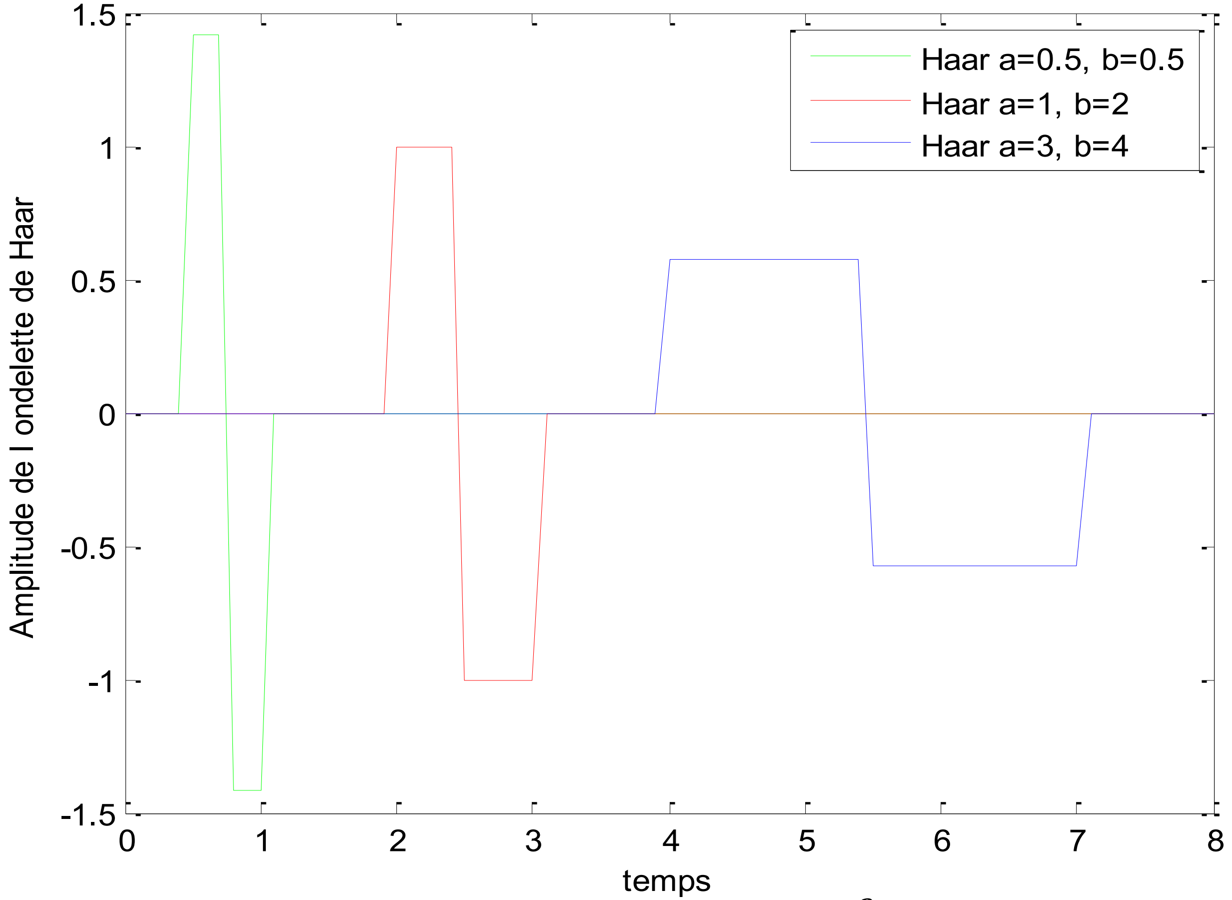
The mother wavelet $\psi(t)$ must have good localization (zero outside a certain interval), and must be oscillating (the number of zero moments corresponds to the number of oscillations).

# IV. Continuous Wavelets

The CWT is a linear operator, invariant by translation and by dilation. It is unique for a given wavelet.

**Example:** Haar

- $\psi(t) = \begin{cases} 1 & si \quad 0 \le t \le \frac{1}{2} \\ -1 & si \quad \frac{1}{2} \le t \le 1 \\ 0 & ailleurs \end{cases}$
- $\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) = \begin{cases} \frac{1}{\sqrt{a}} & si \quad b \le t \le b+\frac{a}{2} \\ -\frac{1}{\sqrt{a}} & si \quad b+\frac{a}{2} \le t \le b+a \\ 0 & ailleurs \end{cases}$



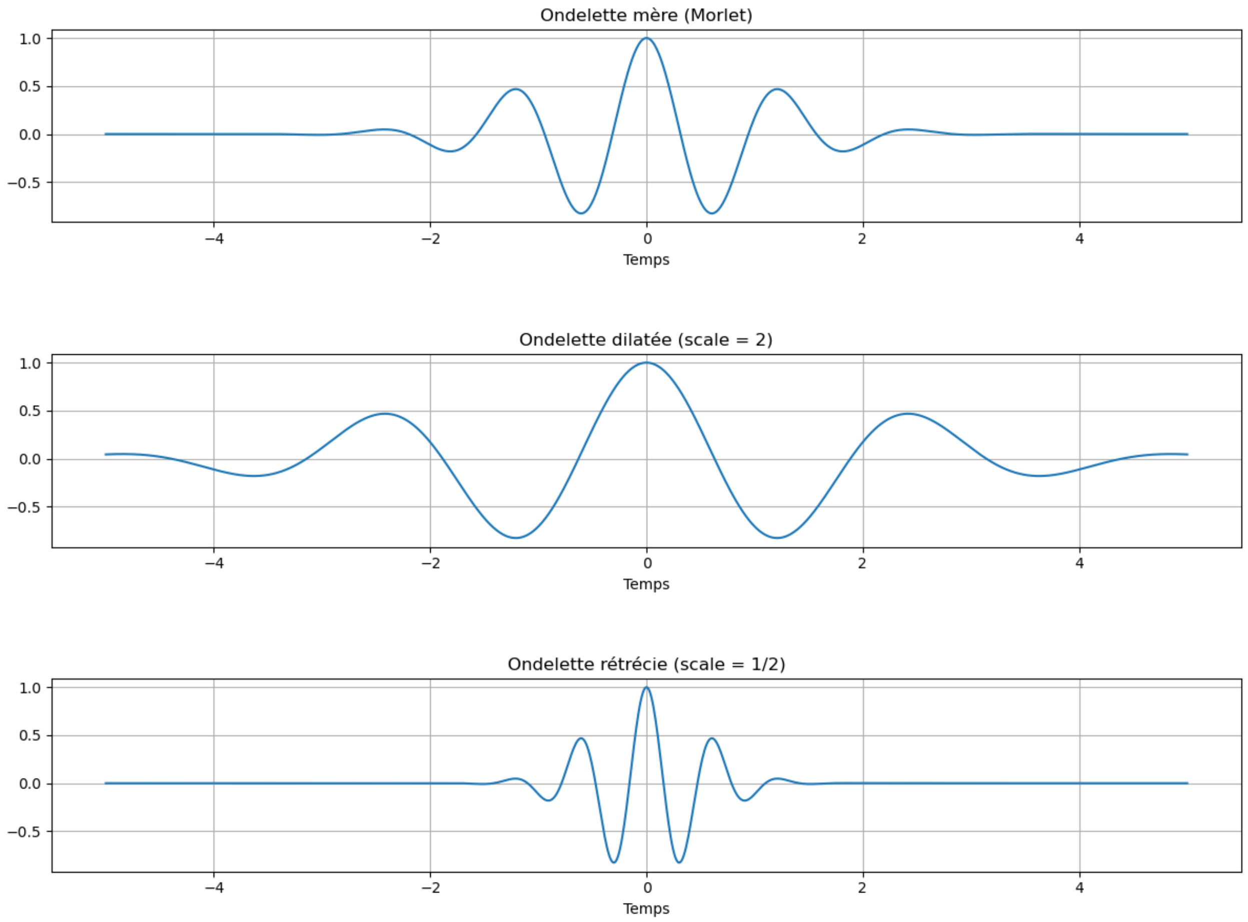

Conservation of ψ a,b is $\|\psi_{a,b}\| = \int_{-\infty}^{+\infty} \frac{1}{a}\left|\psi\left(\frac{t-b}{a}\right)\right|^2 dt = \|\psi\|^2 \quad \forall a$

**Examples of other Wavelets**

- Mexican hat $\psi(t) = \frac{1}{\sigma\sqrt{2\pi}}\left(1-\frac{t^2}{\sigma^2}\right)e^{-\frac{t^2}{2\sigma^2}}$
- Morlet $\psi(t) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}}e^{-2\pi jft}$

# IV. Continuous Wavelets

Wavelet analysis begins with an analysis window ***a*** of very fine width, translates it over the entire signal then starts again by increasing the scale.

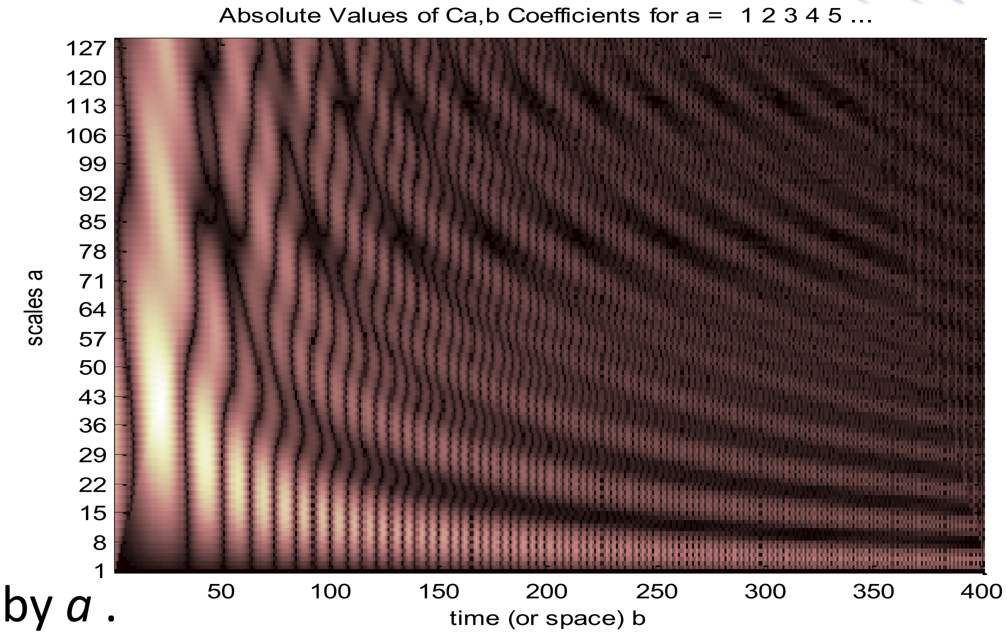



- Its coefficients measure, in a certain sense, the fluctuations of the signal $x(t)$ around the point $t = b$ on the scale provided by $a$ .

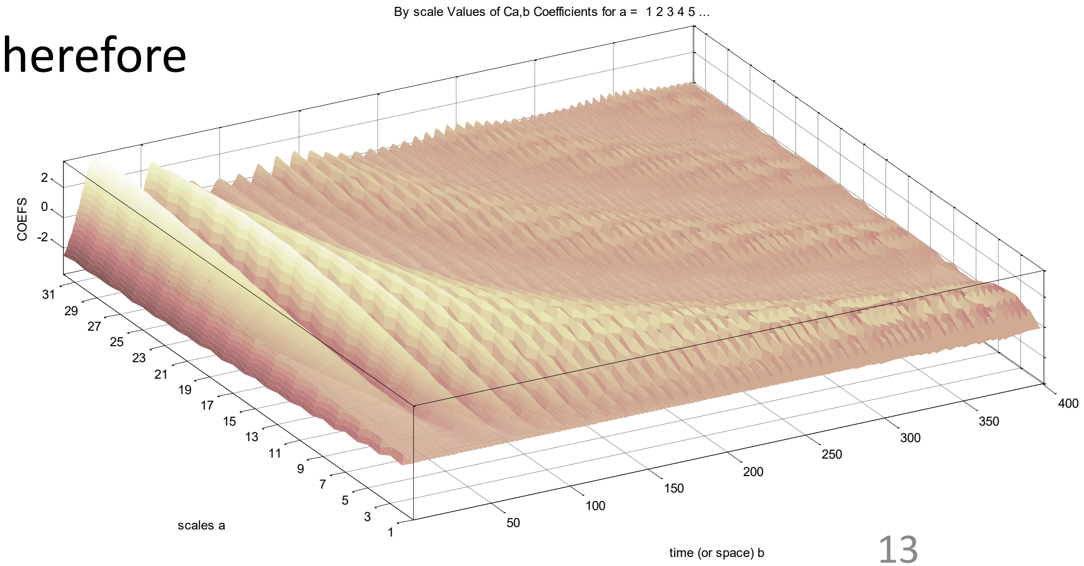
- By decreasing $a$ , the support of $\psi_{a,b}$ reduced in Time and therefore covers a larger frequency range and vice versa.

- So $1/a$ is proportional to a frequency.

# IV. Continuous Wavelets




chirp signal with Haar wavelet




- Wavelet analysis starts with a window of analysis ***a*** of very fine width, translates it over everything the signal then starts again by increasing the scale.

- Its coefficients measure, in a certain sense, the fluctuations of the signal $x(t)$ around the point $t = b$ , at the scale provided by $a$ .

- By decreasing $a$ , the support of $\psi_{a,b}$ reduced in time and therefore covers a larger frequency range and vice versa.

- a $\uparrow \Rightarrow$ general form, has $\downarrow \Rightarrow$ singularities

- Redundancies $\Rightarrow$ Discrete wavelets

# V. Dyadic Discrete Wavelets

Large amount of information to process ⇒ Optimize calculations and data size.

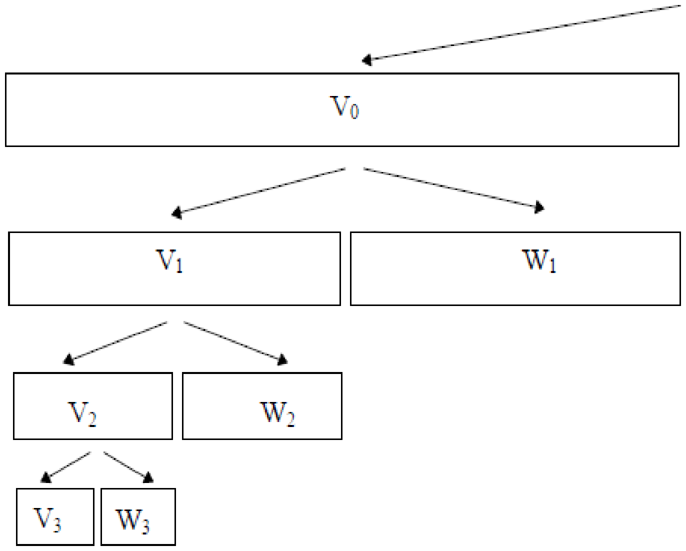
- The multiresolution analysis developed by Meyer and Mallat allows to analyze a signal at different scales through linear operators at resolution levels corresponding to different spatial frequency bands.
- multiresolution analysis using subspaces Vj nested within each other, such that the passage from one to the other is the result of a change of scale (zoom).
- In the dyadic case: we will have a dilation of 2, the space Vj+1 contains signals

more "coarse" than the Vj space → Vj+1 ⊂ Vj

Noticed :

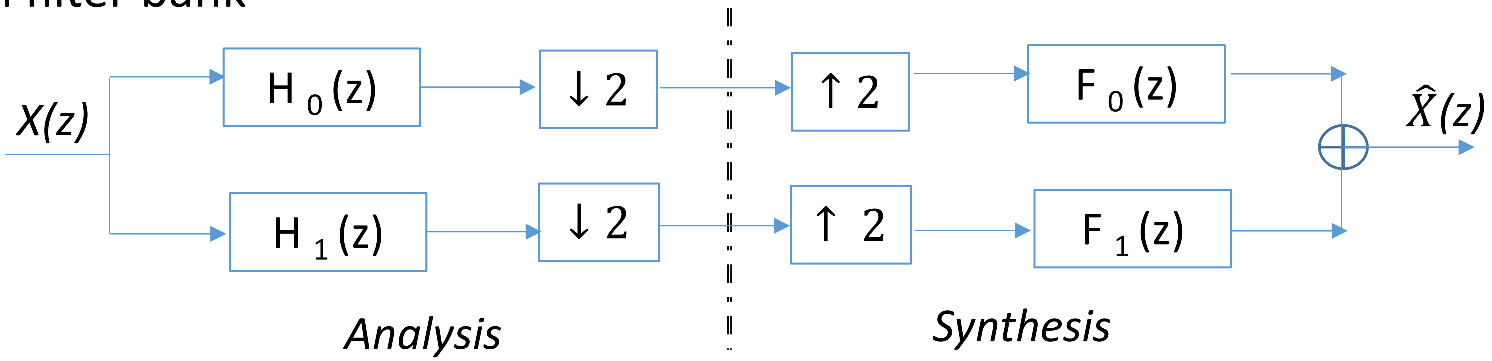
- The construction of such wavelets can be approached as

a problem of decomposition into sub-bands → It lends itself well to a decomposition/reconstruction by filter bank

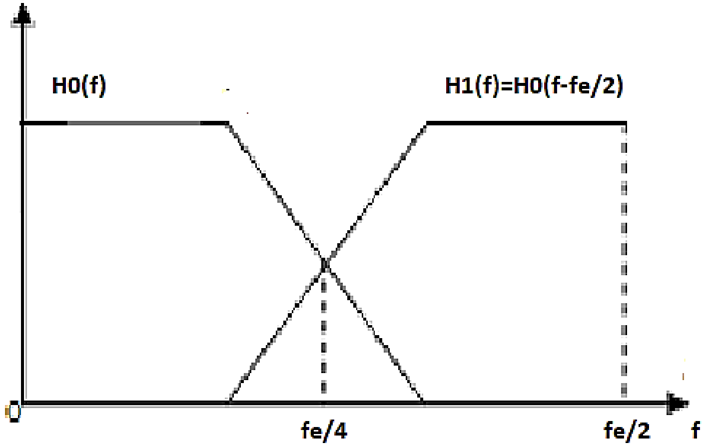


# Filter Bank Reminders

**Example** : 2-channel filter bank




- ✓ $H_0(z)$ and $H_1(z)$ the low-pass and high-pass filters of the **analysis bench**
- ✓ $F_0(z)$ and $F_1(z)$ the low-pass and high-pass filters of the **synthesis bench**

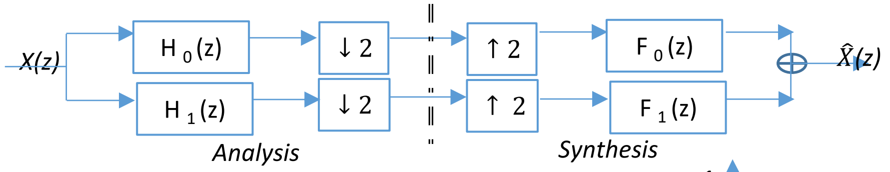


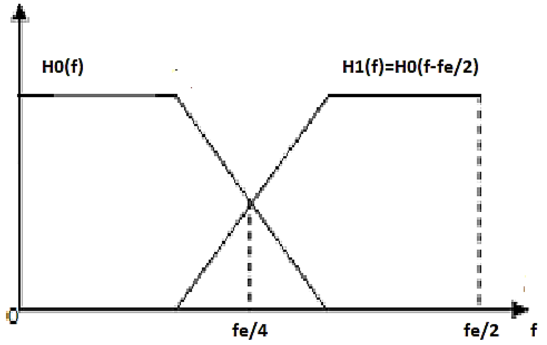

**Critical sampling**

$$H_1(z) = H_0(-z) \Rightarrow H_1(f) = H_0\left(f - \frac{f_e}{2}\right)$$

$$H_1(z) = H_0(-z^{-1}) \Rightarrow H_1(f) = H_0\left(\frac{f_e}{2} - f\right)$$

# Filter Bank Reminders

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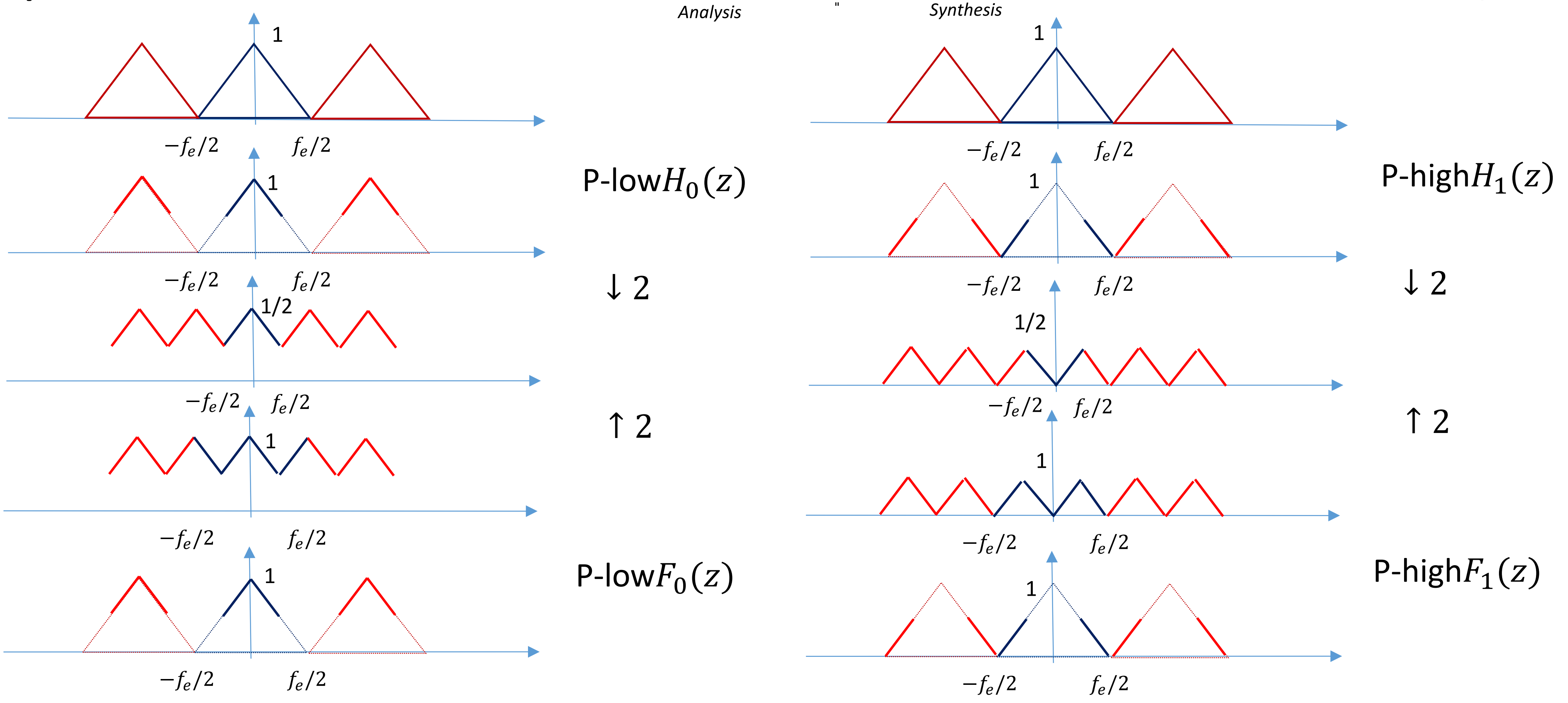
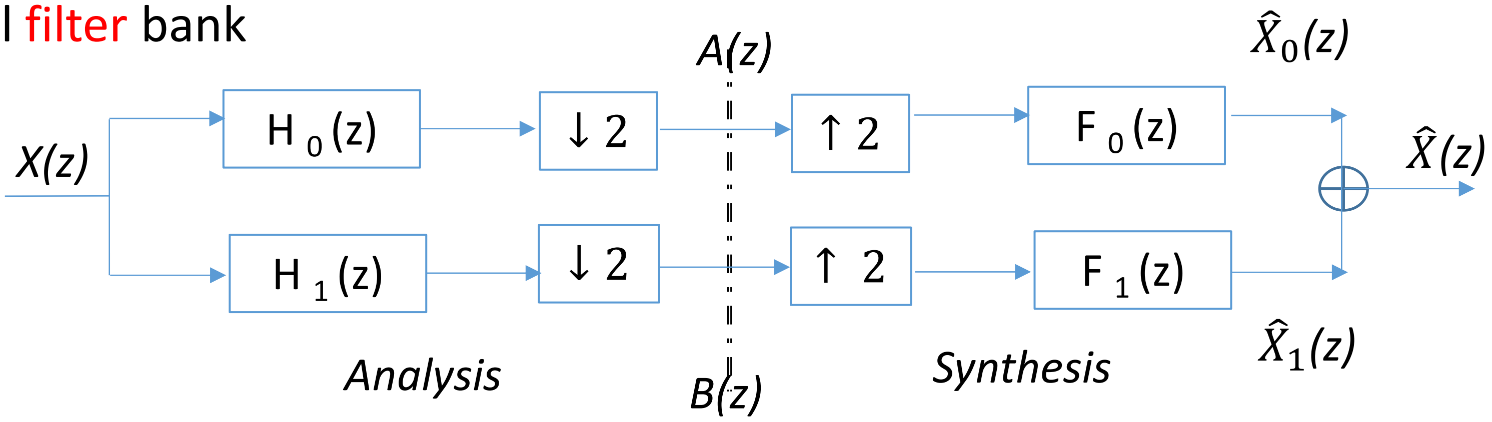


# Filter Bank Reminders

**Example :** 2-channel filter bank




- $A(z) = \left(\frac{1}{2}\sum_{k=0}^{1} H_0\left(W_2^k z^{\frac{1}{2}}\right) X\left(W_2^k z^{\frac{1}{2}}\right)\right)$with $W_2^1$=$e^{-\frac{2\pi j}{2}} = -1$

$\Rightarrow A(z) = \frac{1}{2}\left(H_0\left(z^{\frac{1}{2}}\right)X\left(z^{\frac{1}{2}}\right) + H_0\left(-z^{\frac{1}{2}}\right)X\left(-z^{\frac{1}{2}}\right)\right)$

- $\hat{X}_0(z) = A(z^2)F_0(z) \Rightarrow \hat{X}_0(z) = F_0(z)\frac{1}{2}\left(H_0(z)X(z) + H_0(-z)X(-z)\right)$
- Likewise$\hat{X}_1(z) = F_1(z)\frac{1}{2}\left(H_1(z)X(z) + H_1(-z)X(-z)\right)$

$\Rightarrow \boldsymbol{\hat{X}(z) = \frac{1}{2}\left(F_0(z)H_0(z) + F_1(z)H_1(z)\right)X(z) + \frac{1}{2}\left(F_0(z)H_0(-z) + F_1(z)H_1(-z)\right)X(-z)}$

# V. Dyadic Discrete Wavelets

$$\boldsymbol{F_0(z)H_0(-z) + F_1(z)H_1(-z) = 0}$$

Perfect reconstruction $\Rightarrow$ Quadrature conjugate mirror filters .

$H_1(z) = H_0(-z^{-1})\, z^{-(L-1)}$ or $h_1(n) = -(-1)^n h_0(L-1-n)$

$F_0(z) = -H_1(-z) = H_0(z^{-1})\, z^{-(L-1)}$ i.e. $f_0(n) = h_0(L-1-n)$

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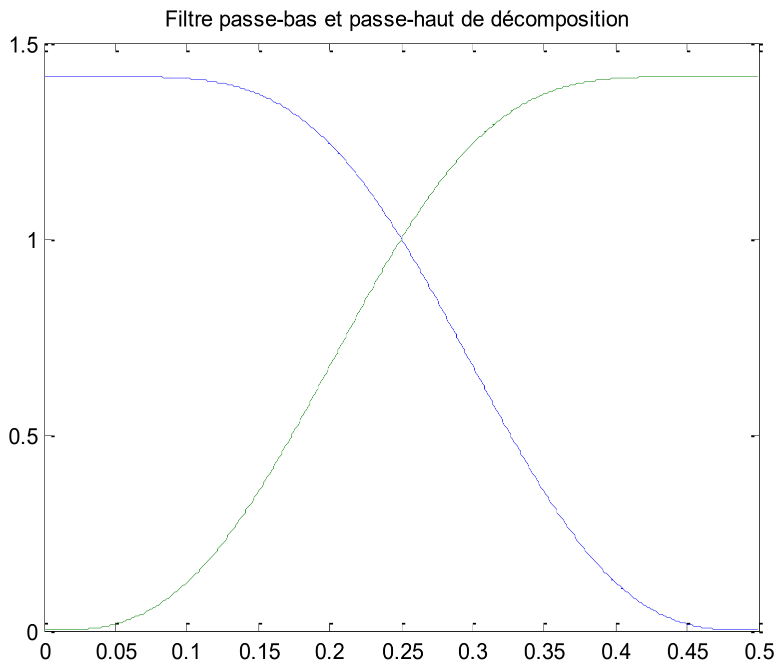



**Example 1**

$$H_0(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}$$

$$\Rightarrow\ H_1(z) = (b_0 - b_1 z^1 + b_2 z^2 - b_3 z^3)z^{-3} = -b_3 + b_2 z^{-1} - b_1 z^{-2} + b_0 z^{-3}$$

$$\Rightarrow\ F_0(z) = (b_0 + b_1 z^1 + b_2 z^2 + b_3 z^3)z^{-3} = b_3 + b_2 z^{-1} + b_1 z^{-2} + b_0 z^{-3}$$

$$\Rightarrow\ F_1(z) = -(b_0 - b_1 z^{-1} + b_2 z^{-2} - b_3 z^{-3}) = b_0 - b_1 z^{-1} + b_2 z^{-2} - b_3 z^{-3}$$

The coefficients are not symmetrical, the phase shift will not be linear. The only orthogonal wavelet that is symmetrical is the Haar wavelet .

- They intersect at *fe* /4. In this case, we can use a decimator to reduce the amount of information by a factor of 2

# V. Dyadic Discrete Wavelets

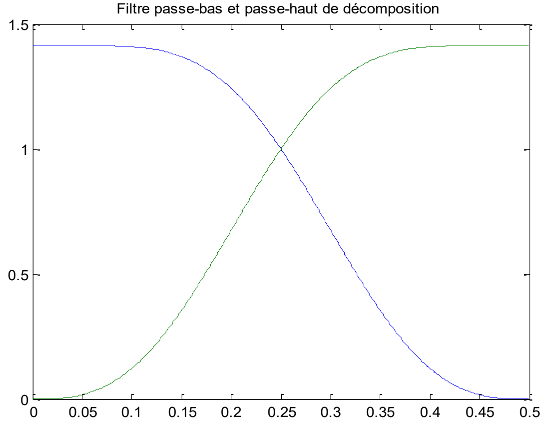
What Filters??

The wavelet must be very compact thus → its coefficients must be, for the most part, close to zero.

(regularity of the function, the number of zero moments and the size of its support)

→ The compactness of the support requires that the filters ($h_0$) which will generate the scale function (giving a rough approximation of the signal) and the wavelet ($h_1$ providing the details) have a finite impulse response.

- Examples of Finite Impulse Response Filters : Daubechies , Haar




**Perfect reconstruction if using CQF filters**

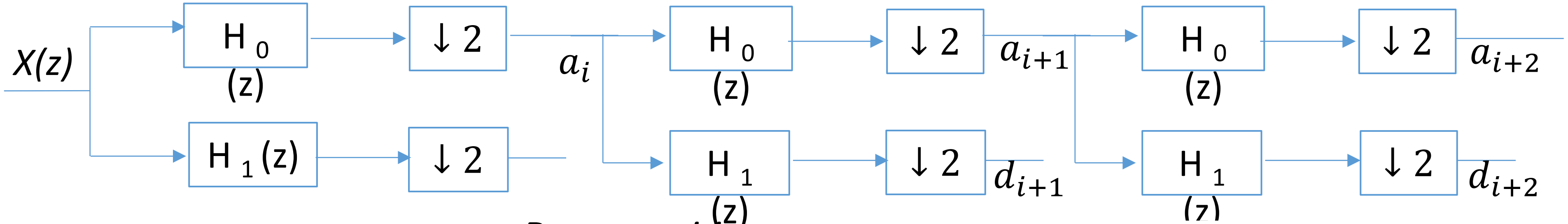
- The term quadrature comes from the fact that the sum of the squared moduli

of the two filters is constant (2 or 1) called the orthogonality condition

- $|H_0(f)|^2 + |H_0(f \pm f_e/2)|^2 = 2, |H_1(f)|^2 + |H_1(f \pm f_e/2)|^2 = 2$
- $|H_0(f)|^2 + |H_1(f)|^2 = 2 \quad H_0(f)H_1^*(f) + H_0(f \pm f_e/2)H_1^*(f \pm f_e/2) = 0$

# V. Dyadic Discrete Wavelets

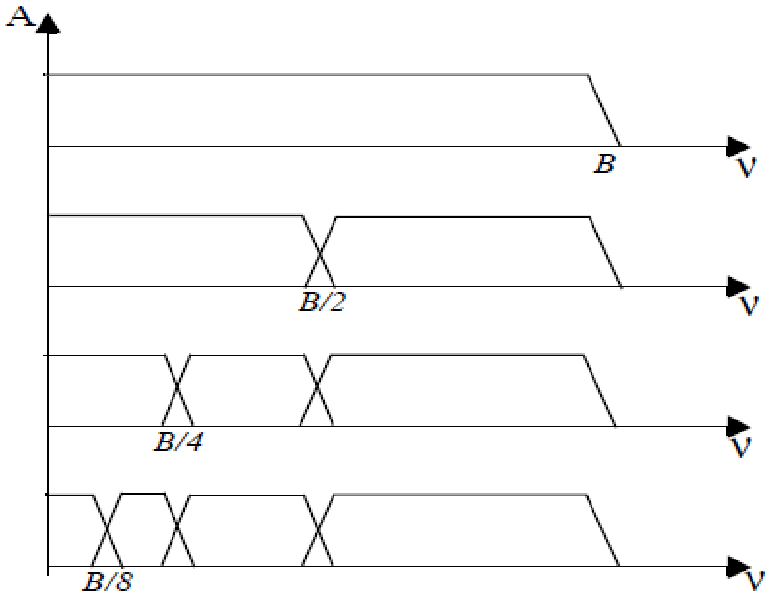
- The construction of such wavelets can be approached as a problem of choice of signal decomposition basis, but also as a problem of decomposition into sub-bands.
- It lends itself well to decomposition/reconstruction by filter bank

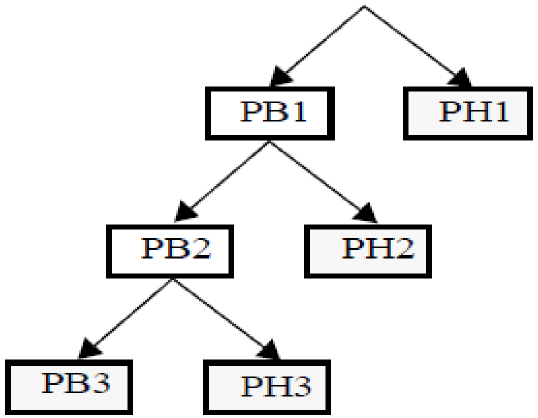



*Decomposition*

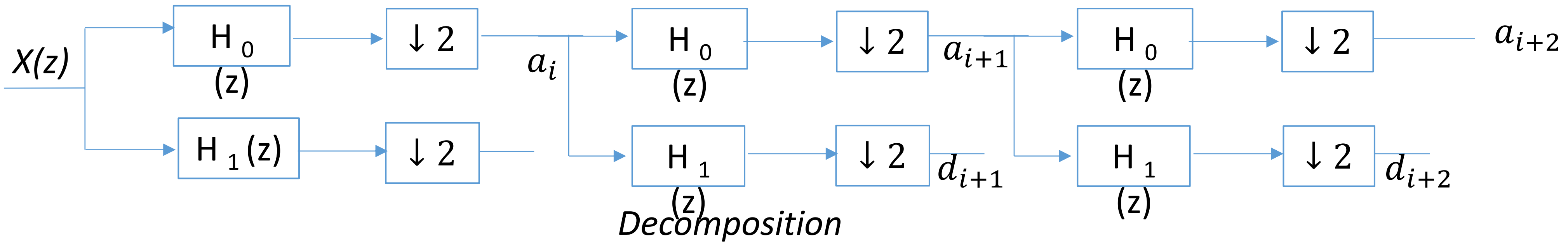
$$a_{i+1}[n] = \sum_k h_0[2n-k]a_i[k]$$

$$d_{i+1}[n] = \sum_k h_1[2n-k]a_i[k]$$

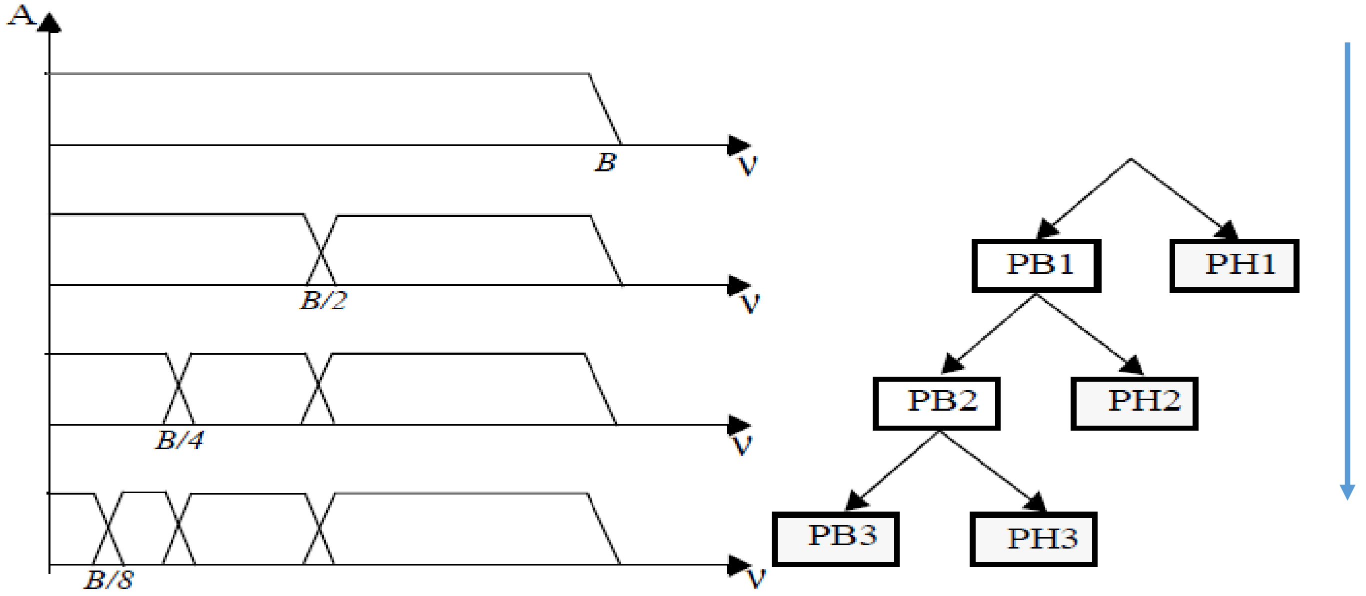


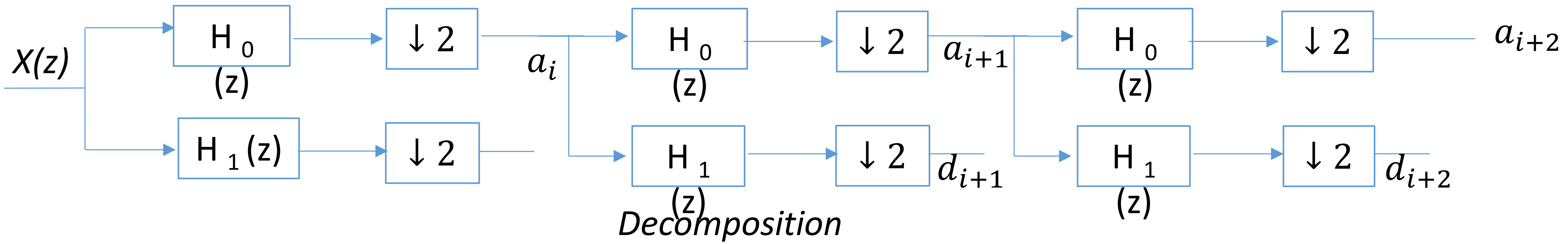
# V. Dyadic Discrete Wavelets




- The low-pass filter (scale function) gives the coarse information while the high-pass filter (wavelet) encodes the details → only 2 filters
- The discrete wavelet transform on orthonormal bases is reduced to digital filtering operations followed by subsampling



# V. Dyadic Discrete Wavelets

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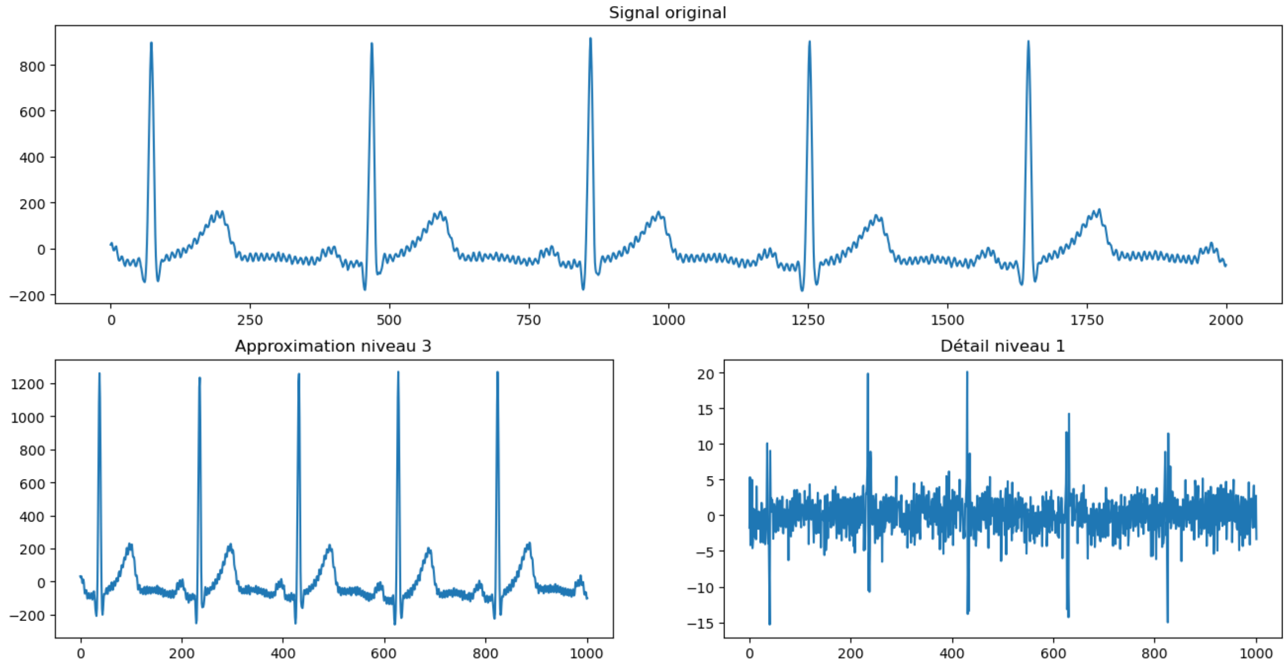
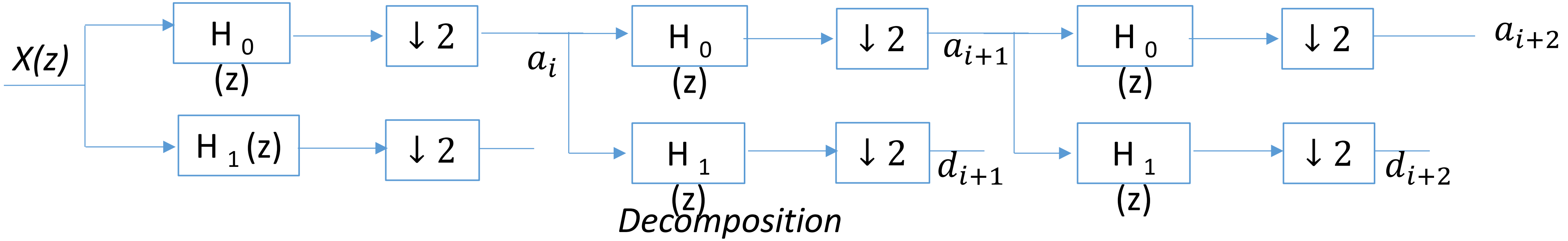


# V. Dyadic Discrete Wavelets

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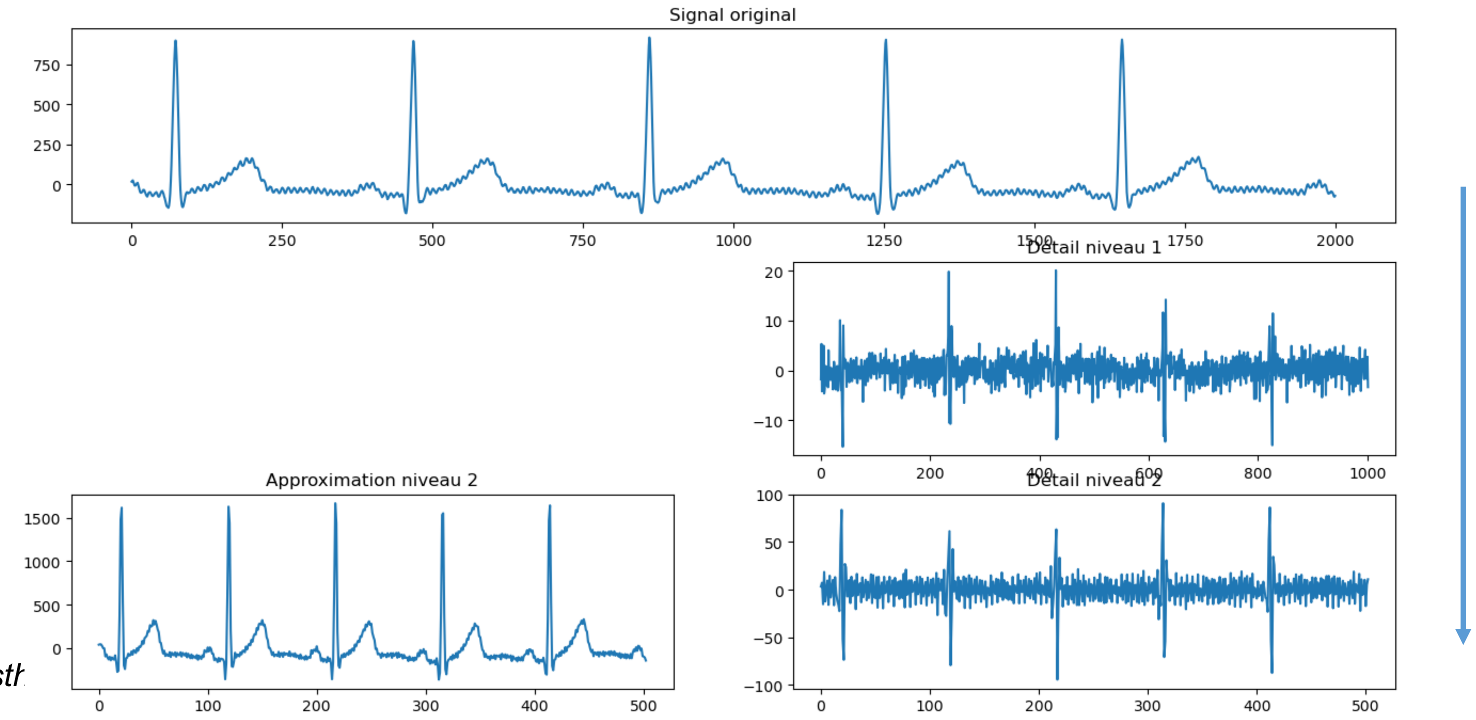
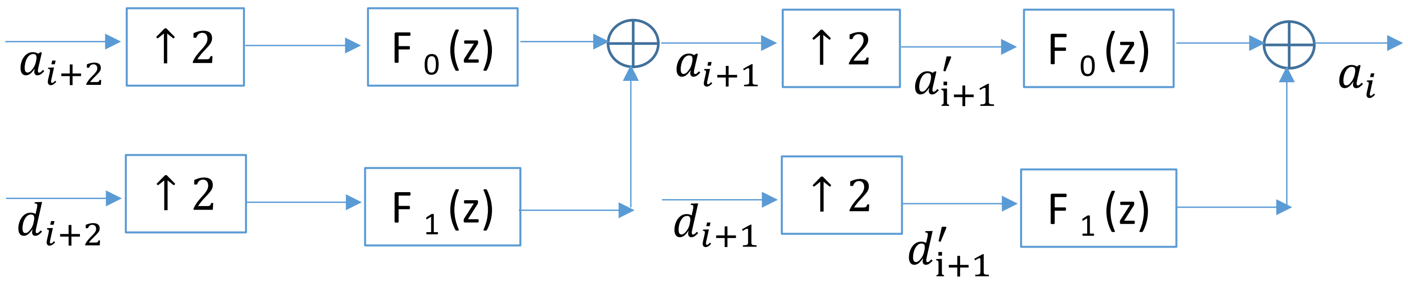


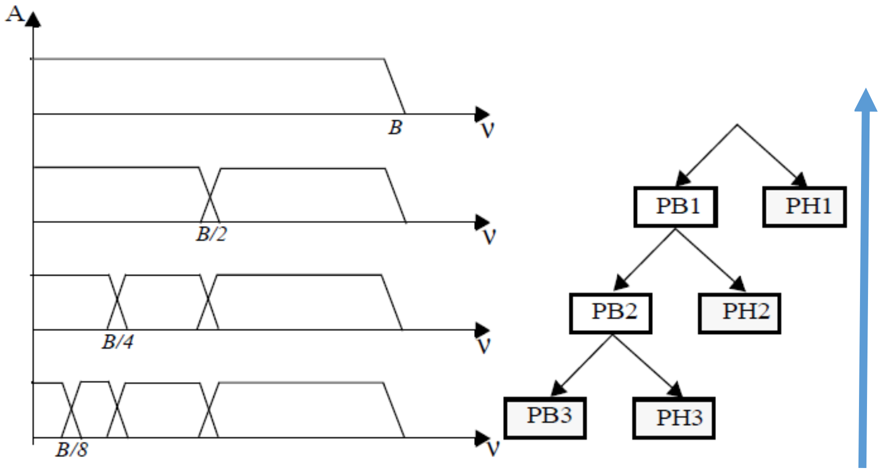
# V. Dyadic Discrete Wavelets

Reconstruction

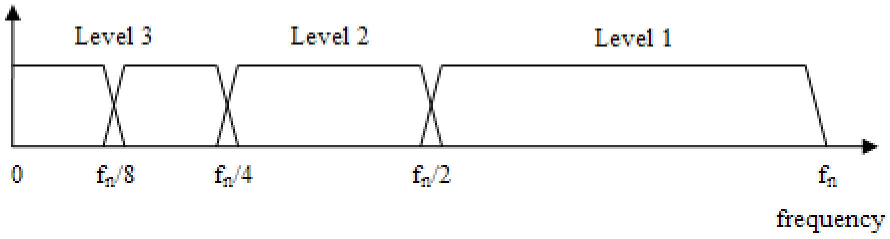
$$a_i[n] = \sum_k \{f_0[n-2k]a_{i+1}[k] + f_1[n-2k]d_{i+1}[k]\}$$




*Reconstruction*




$$a_i[n] = \sum_k \{f_0[n-k]a'_{i+1}[k] + f_1[n-k]d'_{i+1}[k]\}$$



# V. Dyadic Discrete Wavelets

Perfect reconstruction $\Rightarrow$ Quadrature conjugate mirror filters .

$H_1(z) = H_0(-z^{-1})\, z^{-(L-1)}$ or $h_1(n) = -(-1)^n h_0(L-1-n)$

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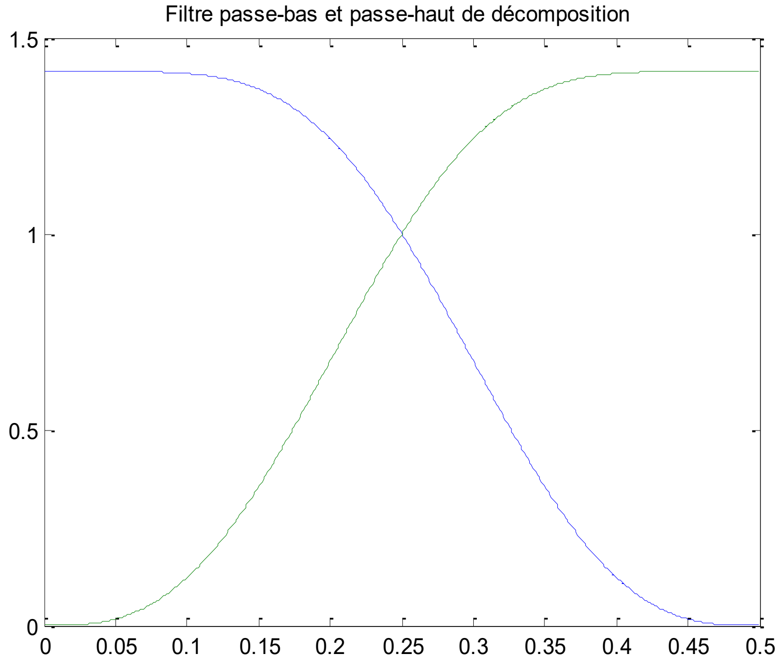


**Example 1**

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$$\Rightarrow F_1(z) = -(b_0 - b_1 z^{-1} + b_2 z^{-2} - b_3 z^{-3}) = b_0 - b_1 z^{-1} + b_2 z^{-2} - b_3 z^{-3}$$

The coefficients are not symmetrical, the phase shift will not be linear. The only orthogonal wavelet that is symmetrical is the Haar wavelet .

- They intersect at *fe* /4. In this case, we can use a decimator to reduce the amount of information by a factor of 2

# V. Dyadic Discrete Wavelets

**Example 2** : Wavelet with 6 coefficients

$h_0$ = [0.2352 0.5706 0.3252 -0.0955 -0.0604 0.0249]

- $h_1$ is obtained by going back $h$ in time and by reversing the sign of odd coefficients

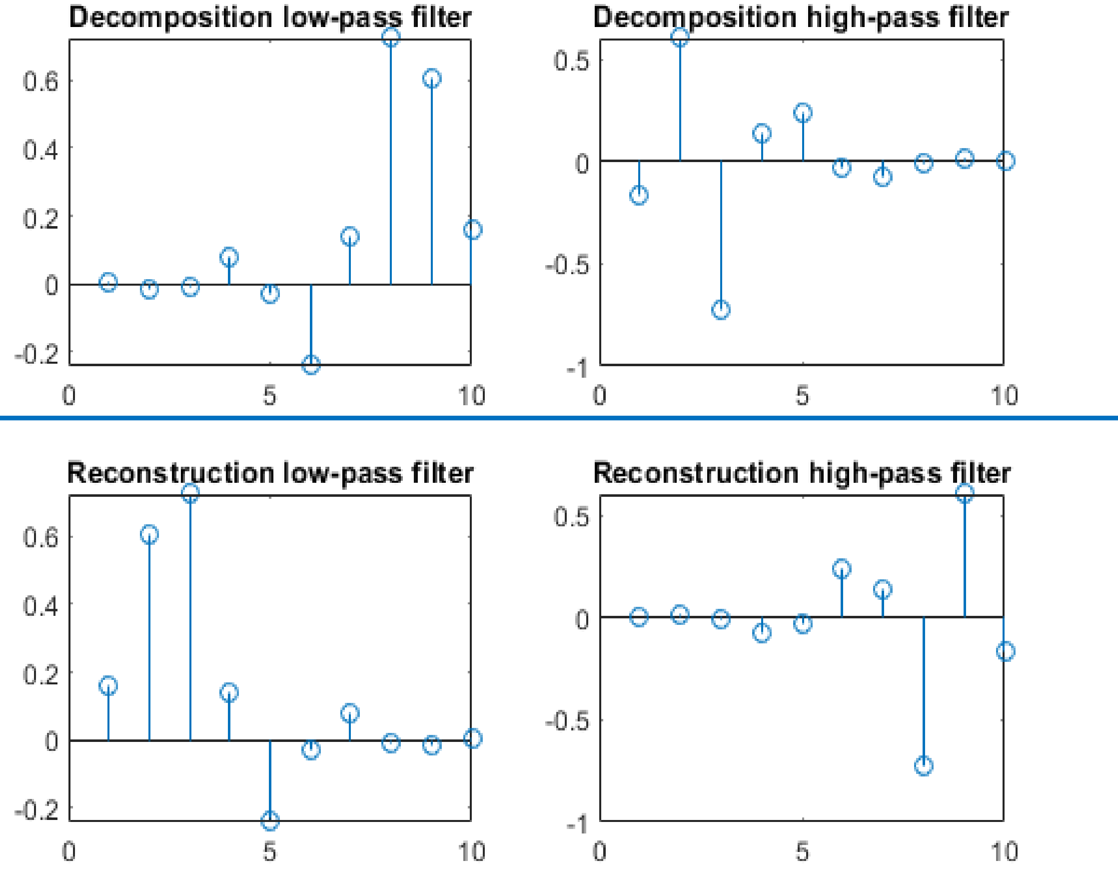
$h_1$ = [-0.0249 -0.0604 0.0955 0.3252 -0.5706 0.2352]

- Reconstruction filters identical to filters analysis but returned in time:

$f_0$ = [0.0249 -0.0604 -0.0955 0.3252 0.5706 0.2352],

$f_1$ = [0.2352 -0.5706 0.3252 0.0955 -0.0604 -0.0249 ]

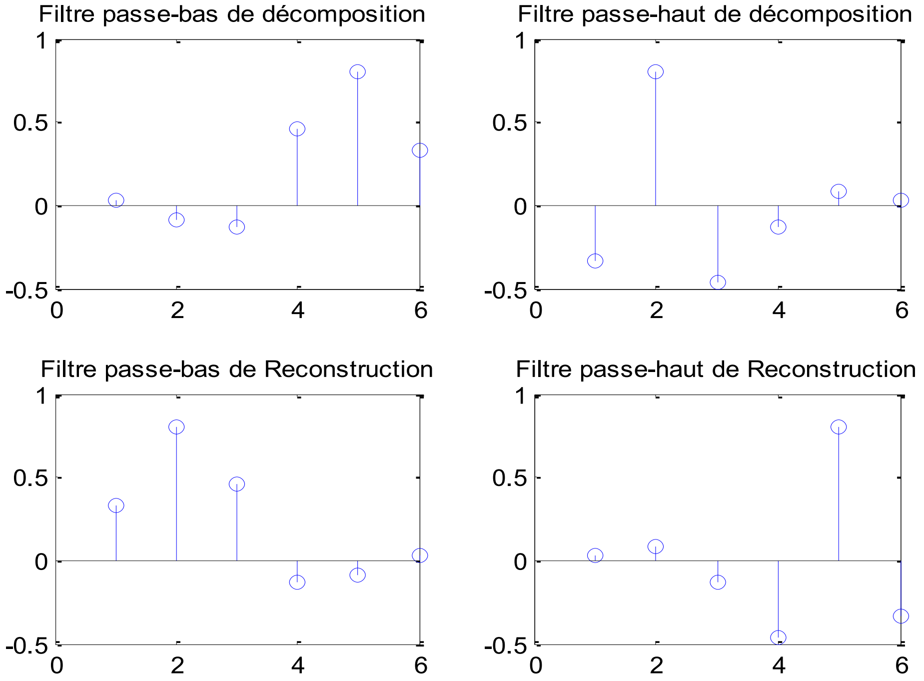
**Example 3**

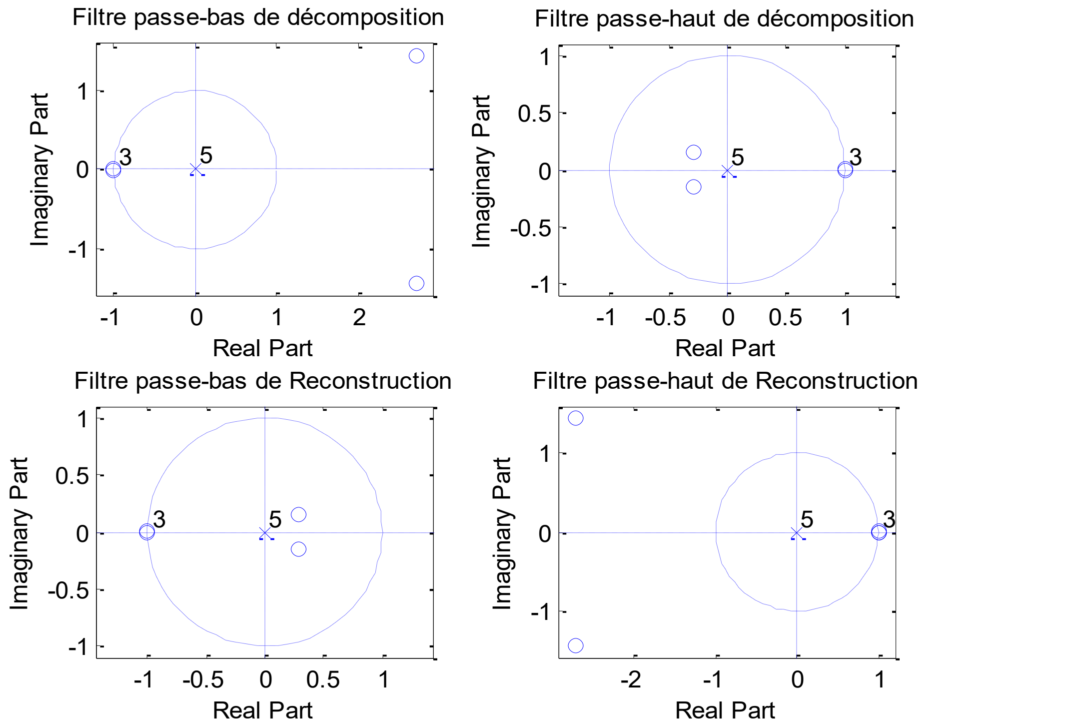


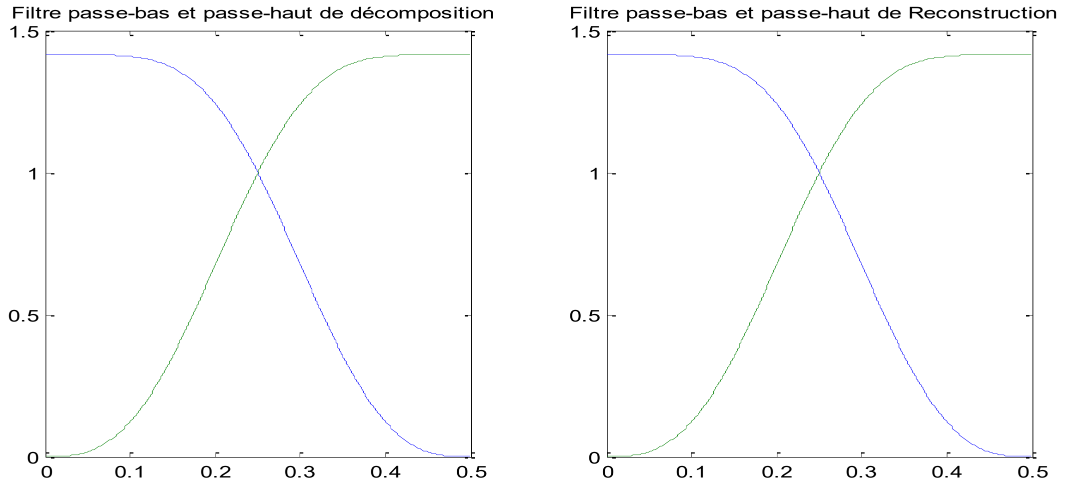

The four filters for db5

# V. Dyadic Discrete Wavelets

## **Example 4** : Daubechies wavelet 3




Daubechies filters lead to response filters

finite impulse response (FIR) but are not linear in phase.

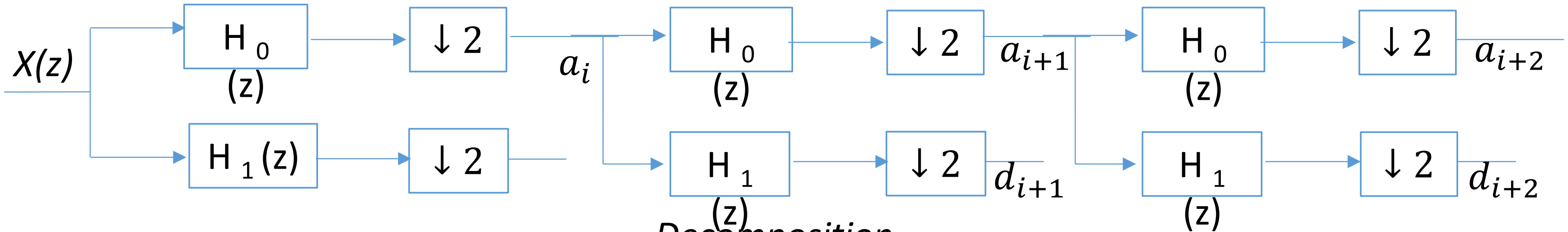
# V. Dyadic Discrete Wavelets

**Application example 1**

We assume that the signal to be analyzed is a ramp and that the wavelet is the Haar wavelet .

**Decomposition** $\boldsymbol{h_0}$Filters = $\left[\frac{1}{\sqrt{2}}\ \ \frac{1}{\sqrt{2}}\right]$, $\boldsymbol{h_1}$= , $\left[-\frac{1}{\sqrt{2}}\ \ \frac{1}{\sqrt{2}}\right]$**Reconstruction** $\boldsymbol{f_0}$Filters = $\left[\frac{1}{\sqrt{2}}\ \ \frac{1}{\sqrt{2}}\right]$, $\boldsymbol{f_1}$=$\left[\frac{1}{\sqrt{2}}\ \ -\frac{1}{\sqrt{2}}\right]$

- Decomposition




*Decomposition*

**Note** we convolve

$$a_{i+1}[n] = \sum_k h_0[2n - k]a_i[k]$$

$$d_{i+1}[n] = \sum_k h_1[2n - k]a_i[k]$$

| | **Approximation** $h_0$[-k]=$\left[\frac{1}{\sqrt{2}}\ \ \frac{1}{\sqrt{2}}\right]$ | **Detail** $h_1$[-k]=$\left[\frac{1}{\sqrt{2}}\ \ -\frac{1}{\sqrt{2}}\right]$ |
|---|---|---|
| **Level 0** | 0,1,2,3,4,5,6,7 | |
| **Level 1** | | |

# V. Dyadic Discrete Wavelets

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We assume that the signal to be analyzed is a ramp and that the wavelet is the Haar wavelet .

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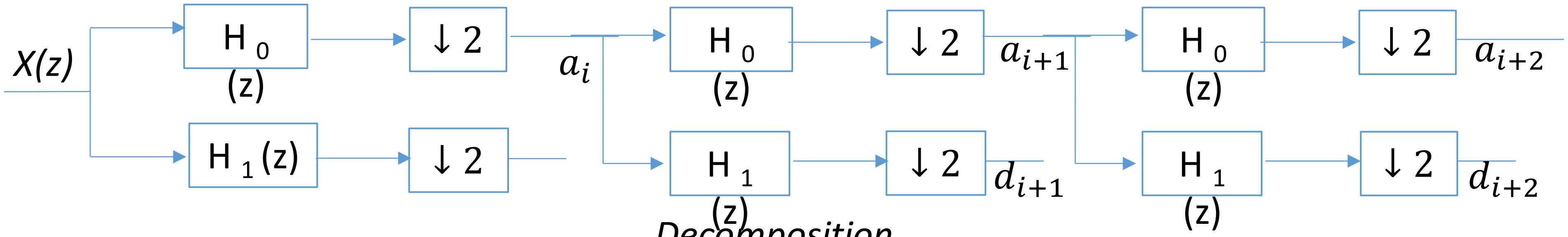
- Decomposition

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|---|---|---|
| **Level 0** | 0,1,2,3,4,5,6,7 | |
| **Level 1** | $\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{9}{\sqrt{2}}, \frac{13}{\sqrt{2}}$ | $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ |

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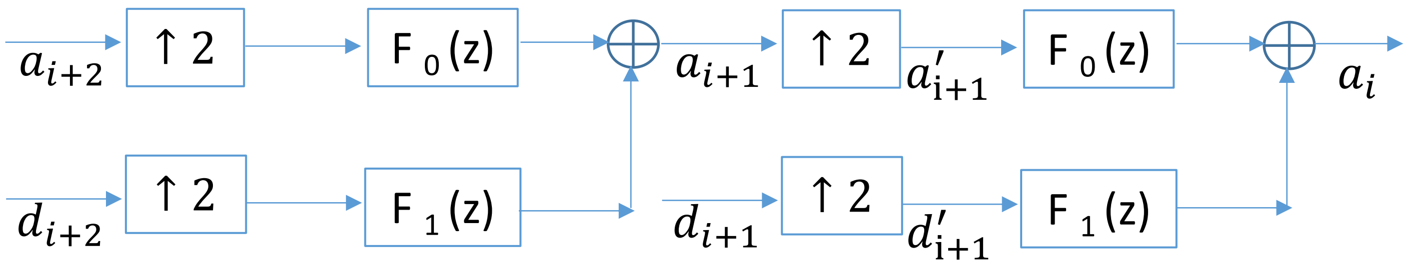
# V. Dyadic Discrete Wavelets

**Application example 1**

We assume that the signal to be analyzed is a ramp and that the wavelet is the Haar wavelet .

**Decomposition $h_0$**Filters = $\left[\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right]$, $h_1$= , $\left[-\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right]$**Reconstruction $f_0$**Filters = $\left[\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right]$, $f_1$=$\left[\frac{1}{\sqrt{2}} \;\; -\frac{1}{\sqrt{2}}\right]$




| | Approximation $f_0[-k]$=$\left[\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right]$ | Detail $f_1$[-k]=$\left[-\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right]$ |
|---|---|---|
| **Level 1** | $\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{9}{\sqrt{2}}, \frac{13}{\sqrt{2}}$<br>$0, \frac{1}{\sqrt{2}}, 0, \frac{5}{\sqrt{2}}, 0, \frac{9}{\sqrt{2}}, 0, \frac{13}{\sqrt{2}}, 0$ | $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$<br>$0, -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0$ |
| **Level 0** | 0,1,2,3,4,5,6,7 | |

# V. Dyadic Discrete Wavelets

**Application example 1**

We assume that the signal to be analyzed is a ramp and that the wavelet is the Haar wavelet .

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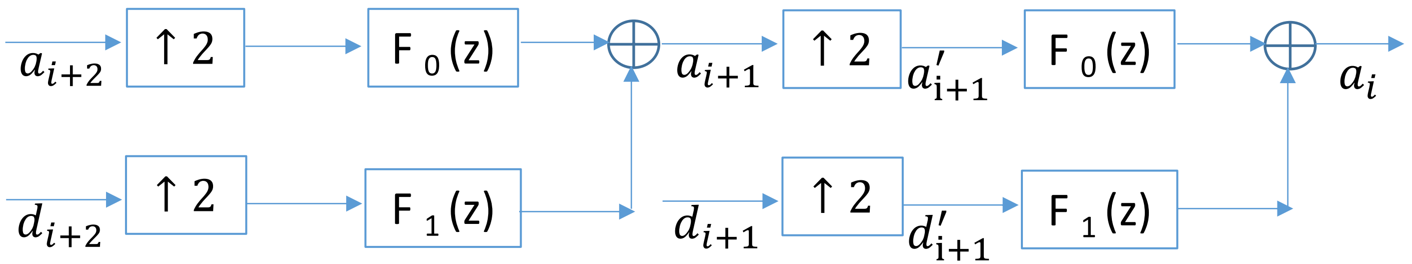


| | Approximation $f_0[-k]$=$\left[\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right]$ | Detail $f_1$[-k]=$\left[-\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right]$ |
|---|---|---|
| **Level 1** | $\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{9}{\sqrt{2}}, \frac{13}{\sqrt{2}}$<br>$0, \frac{1}{\sqrt{2}}, 0, \frac{5}{\sqrt{2}}, 0, \frac{9}{\sqrt{2}}, 0,$<br>$\frac{13}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}, \frac{9}{2}, \frac{9}{2}, \frac{13}{2}, \frac{13}{2}$ | $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$<br>$0, -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0$<br>$-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$ |
| **Level 0** | 0,1,2,3,4,5,6,7 | |

# V. Dyadic Discrete Wavelets

**Application example 2**

Original signal: 11, 9, 5, 7

$h_0$={0.5, 0.5} ⇒ $h_1$={-0.5, 0.5} ,

$f_0$=2*{0.5, 0.5} , $f_1$=2*{0.5, -0.5}

| | Approximation $h_0$[-k]={0.5, 0.5} | Detail $h_1$[-k]={0.5, -0.5} |
|---|---|---|
| Level 0 | 11, 9, 5, 7 | |
| Level 1 | 10, 6 | 1, -1 |
| Level 2 | 8 | 2 |
| Decomposed signal | 8, 2, 1, -1 | |

- Decomposition

$$a_{i+1}[n] = \sum_k h_0[2n - k]a_i[k]$$

$$d_{I+1}[n] = \sum_k h_1[2n - k]a_i[k]$$

| | Approximation $f_0$[-k]={1, 1} | Detail $f_1$[-k]={-1, 1} |
|---|---|---|
| Level 2 | 8<br>0, 8, 0<br>[8, 8 ] + | 2<br>0, 2, 0<br>[2, -2] |
| Level 1 | 10, 6<br>0.10, 0, 6, 0<br>[10,10,6,6 ] + | 1, -1<br>0, 1, 0, -1, 0<br>[1,-1,-1,1] |
| Signal rebuilt | 11, 9, 5, 7 | |

- Reconstruction

$$a_i[n] = \sum_k \{f_0[n - k]a'_{i+1}[k] + f_1[n - k]d'_{i+1}[k]\}$$



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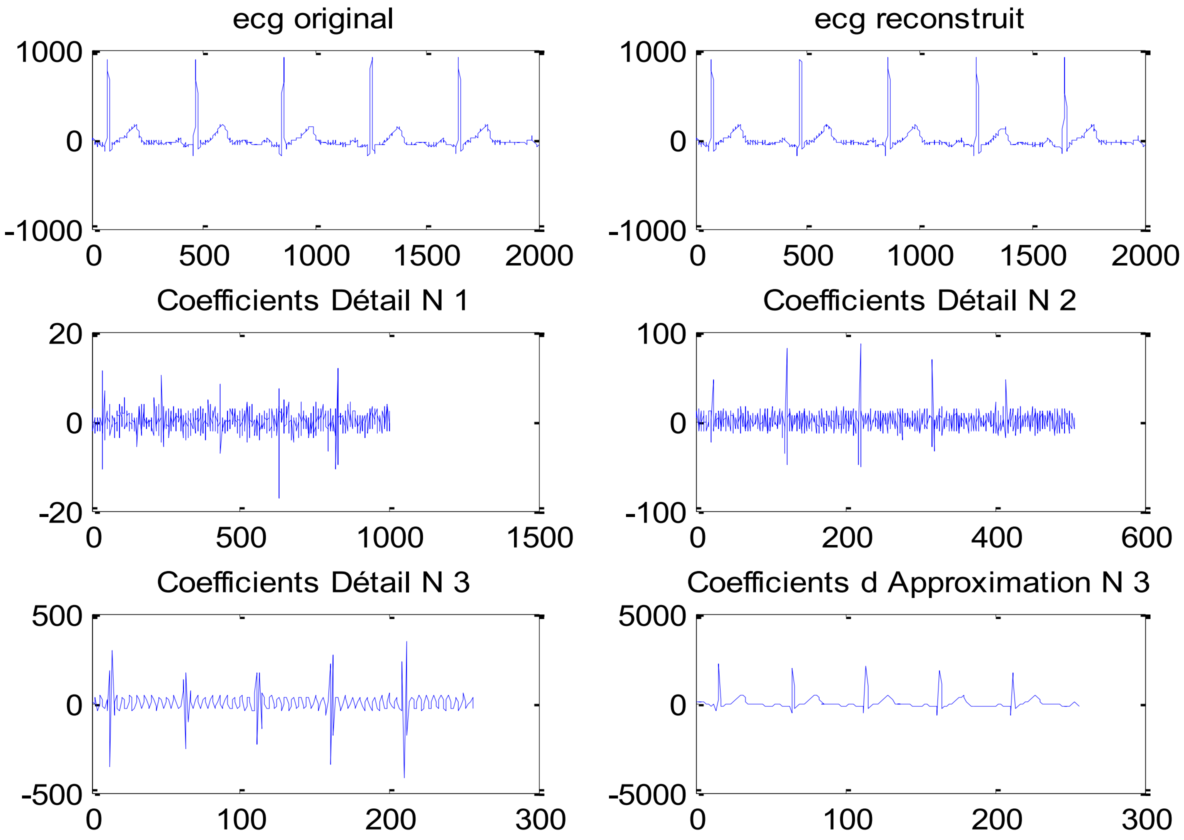
# V. Dyadic Discrete Wavelets

**Application example 3**

- Wavelets are well suited for denoising signals. We decompose the noisy signal and we forces to zero the coefficients that do not pass one threshold. Then we reconstruct the signal




- In the same way we can compress the signal .

- The lifting scheme allows a very simple implementation of wavelet decompositions and their inverse operations by using polyphase factorization.

# V. Dyadic Discrete Wavelets

**Bi-Orthogonal Filters**

Wavelet bases can be constructed leading to linear phase RIF filters such as B splines . These are biorthogonal bases where the decomposition and reconstruction families are different and orthogonal only to each other.

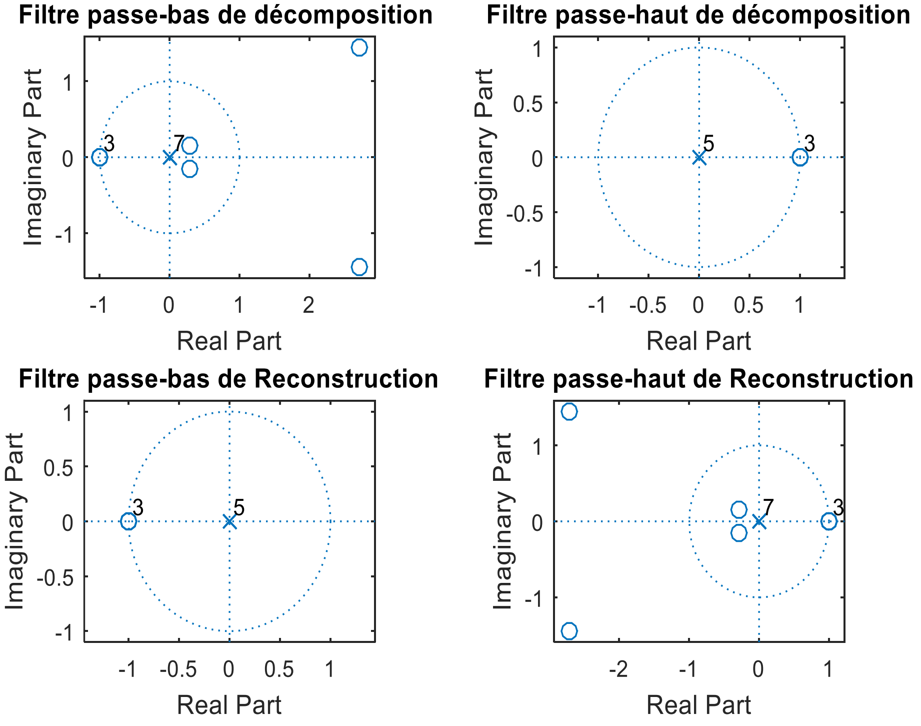
$$H_0(f)F_0^*(f) + H_0(f+f_e/2)F_0^*(f+f_e/2) = 2 \text{And} H_1(f)F_1^*(f) + H_1(f+f_e/2)F_1^*(f+f_e/2) = 2$$

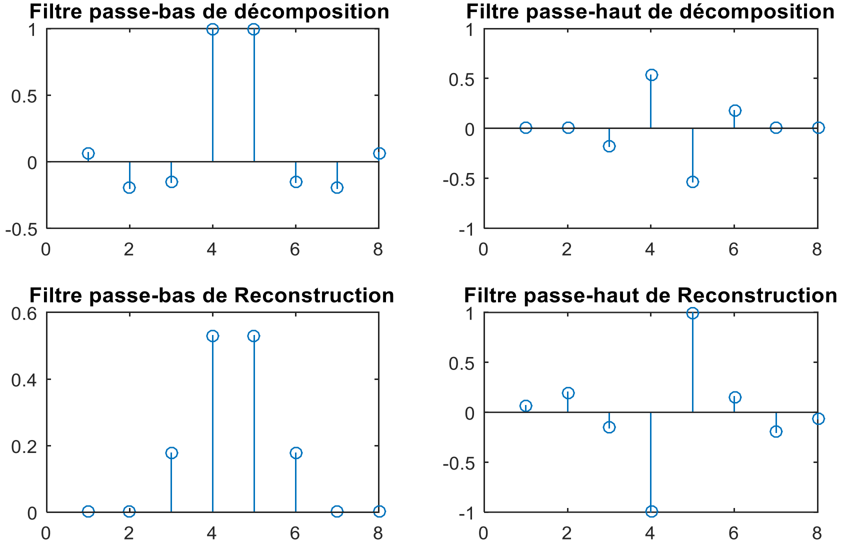
$$H_0(f)F_1^*(f) + H_0(f+f_e/2)F_1^*(f+f_e/2) = 0 \text{And} H_1(f)F_0^*(f) + H_1(f+f_e/2)F_0^*(f+f_e/2) = 0$$

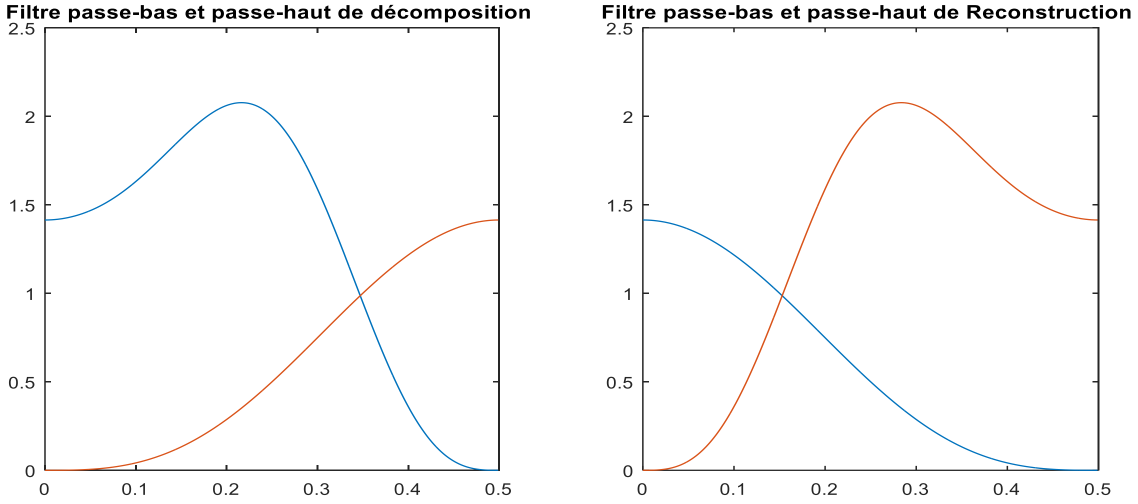
- The decomposition low-pass and high-pass filters are different, as are the reconstruction filters.
- The decomposition and reconstruction filters are dual : The reconstruction high-pass filter $F_1(z)$ is obtained by reversing in time the decomposition low-pass $H_0(z)$ and reversing the sign of one coefficient out of two. Similarly for the reconstruction low-pass filter $F_0(z)$ by reversing in time the decomposition low-pass $H_1(z)$ and reversing the sign of one coefficient out of two, i.e.:
- $h_1(n) =- (-1)^n f_0(L-1-n)$
- $f_1(n) = (-1)^n h_0(L-1-n)$

# V. Dyadic Discrete Wavelets

## Bi-Orthogonal Filters



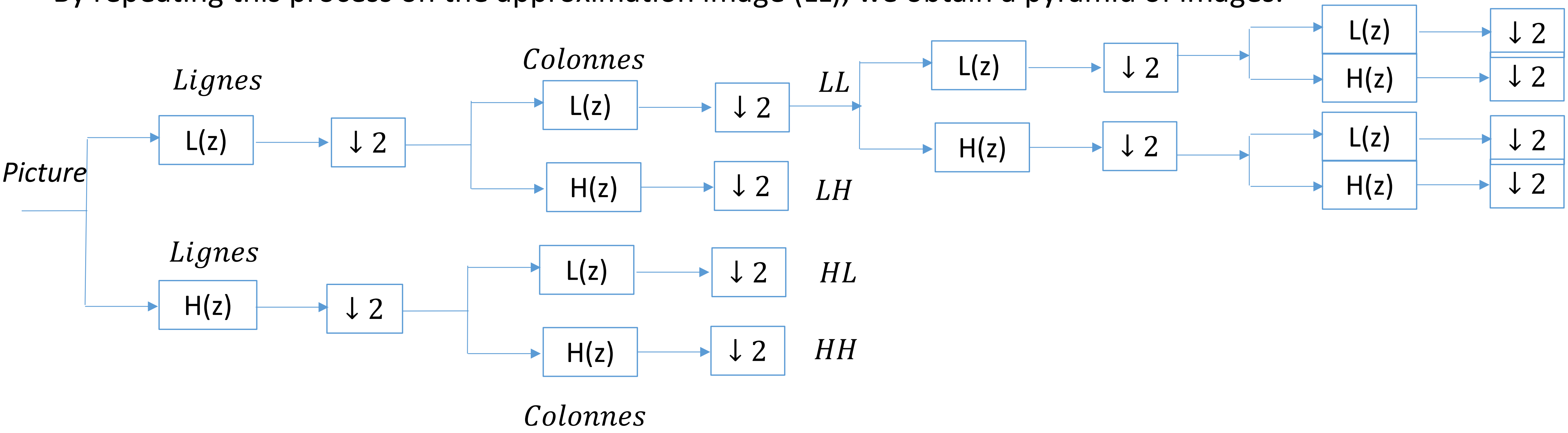



# V. Dyadic Discrete Wavelets

**Example image application**

For an image: 2 filters along the lines, one high-pass (H), the other low-pass (L). Then, we only consider one column out of two and we apply the 2 filters again. By considering only one line out of two, we obtain 4 images of half the size.

By repeating this process on the approximation image (LL), we obtain a pyramid of images.



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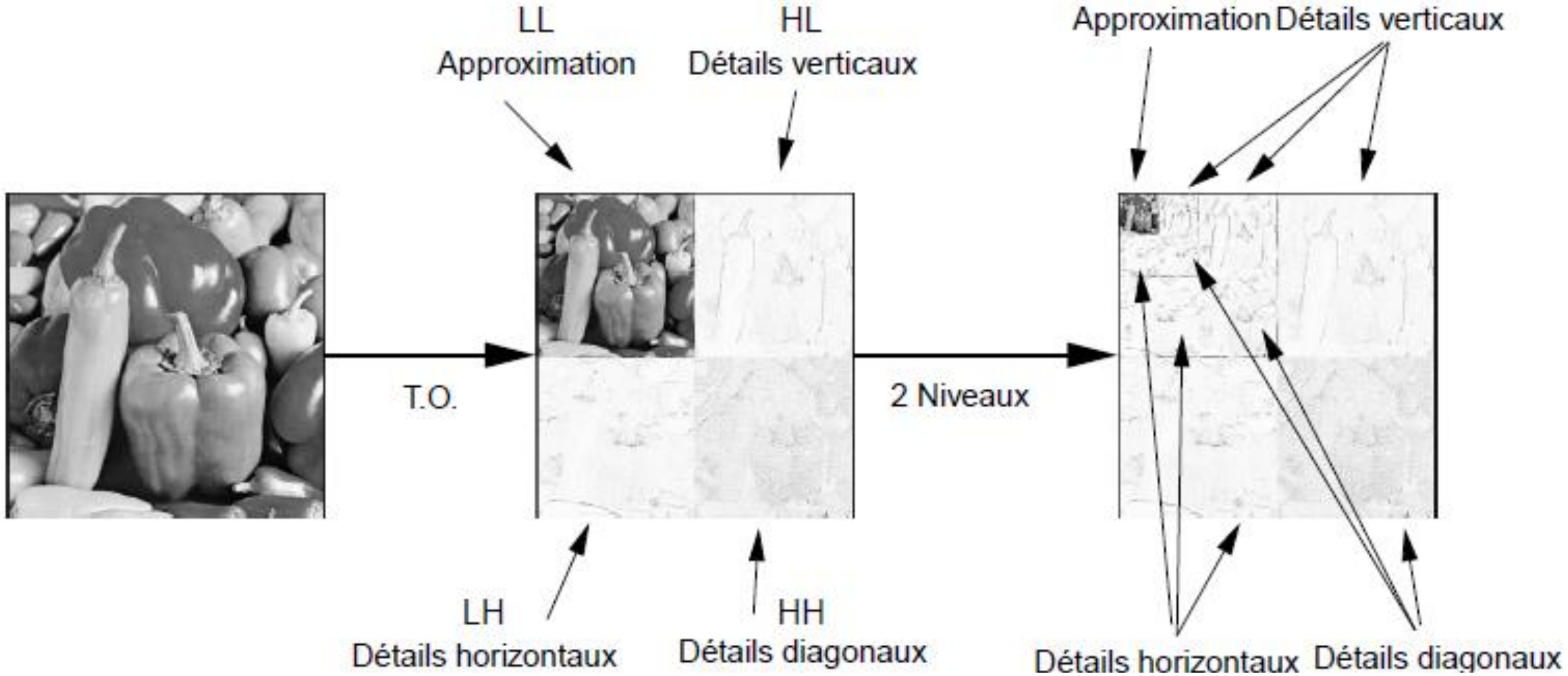

# V. Dyadic Discrete Wavelets

**Example of application in imaging**

In image processing, we prefer bi-orthogonal filters

Image Originale, 24 bpp

jpeg and jpeg2000 (Wavelet) compression with a bpp of 0.2