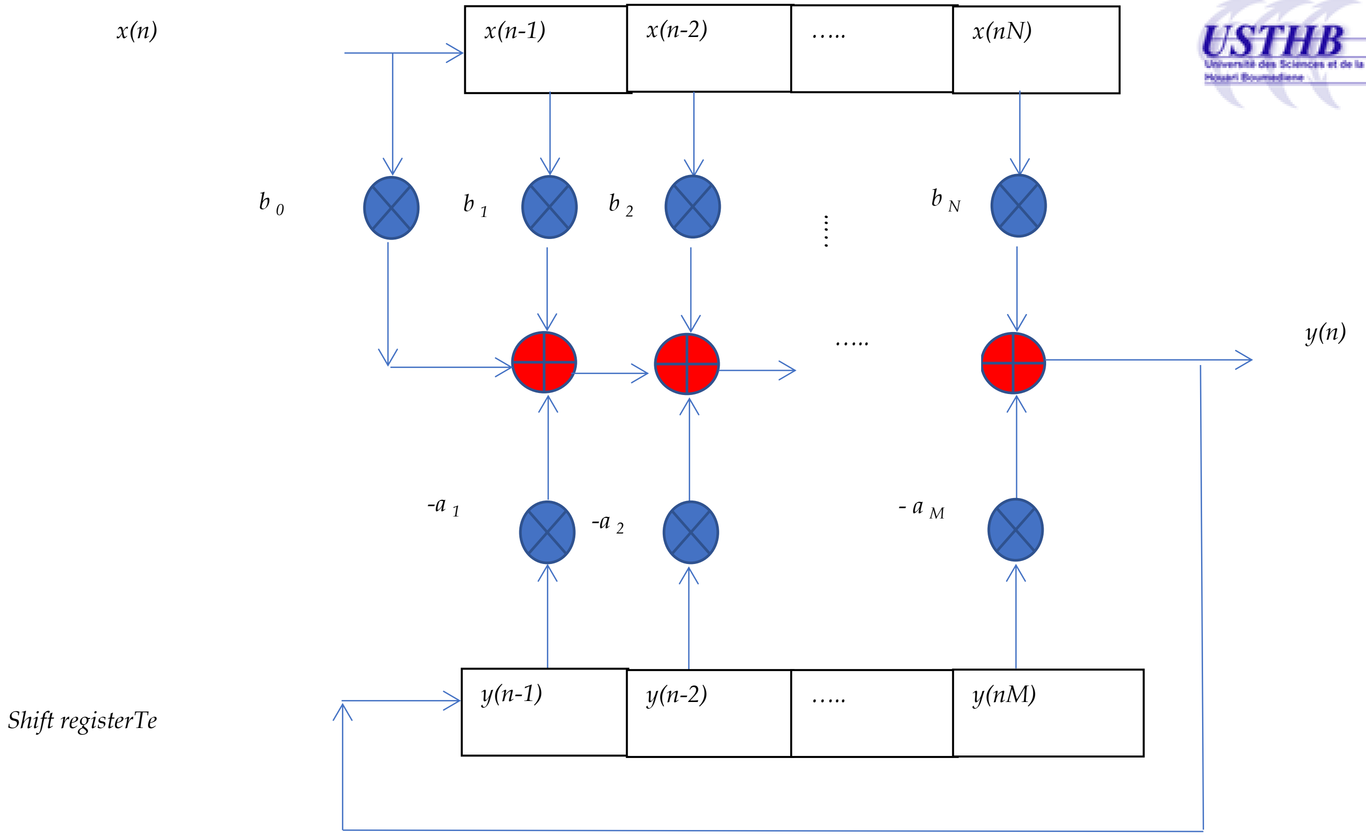
I. Reminders on digital filters (FIR and IIR)

I. Introduction

- Digital Linear Filtering

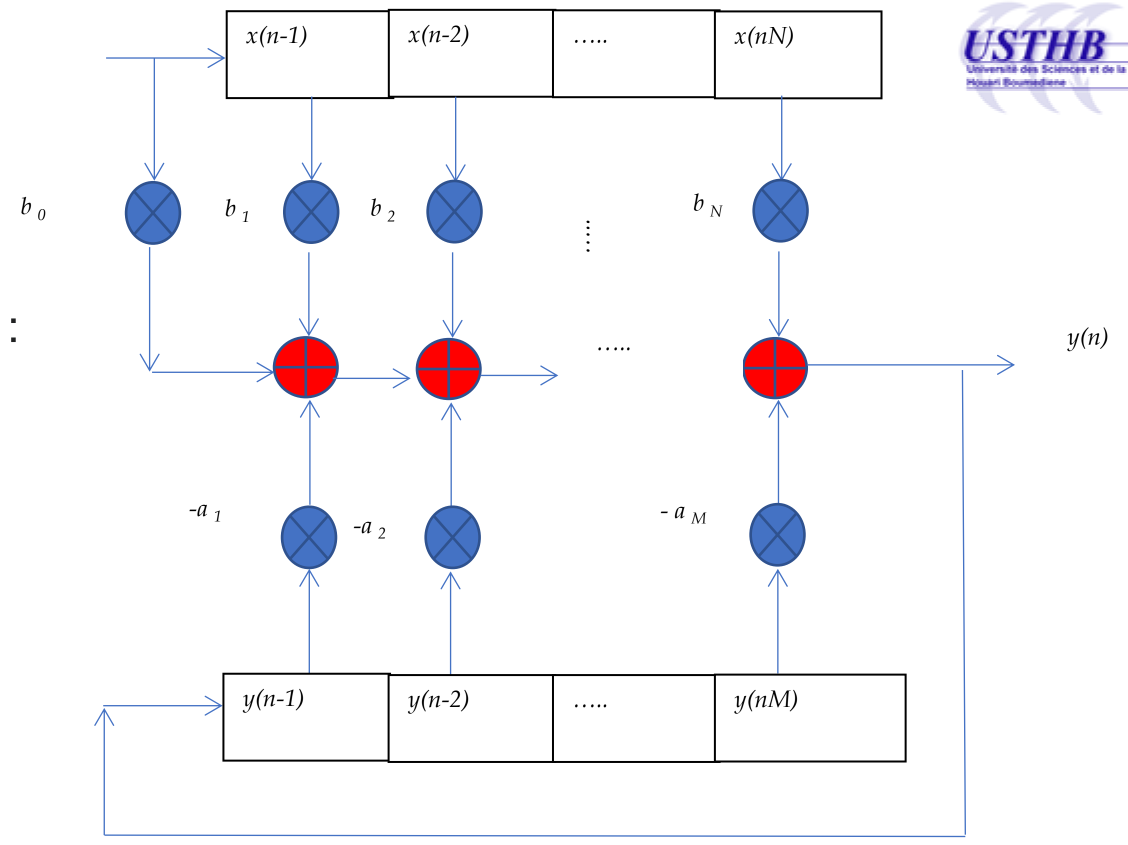
Structure: DLTI

$$y(n) = -\sum_{i=1}^{M} a_i y(n-i) + \sum_{i=0}^{N} b_i x(n-i)$$



- one or more delay devices (shift registers)
- a clock
- arithmetic operators (adders and multipliers)
- registers providing the filter weighting coefficients

I. Introduction



Analog filters consist of **imperfect electronic components** :

Example : Resistor values often have a tolerance of ±5%

May change with temperature and drift with time.

Order of an analog filter ↑ the effect of variable component errors is greatly magnified.

- Digital Linear Filtering
  - Reliable
  - Reproducible
  - Flexible
  - Real time (digital transmissions, MP3 sound coding, speech synthesis, digital television
  - Most analog filter models can thus be reproduced in digital form.

II. Z Transform

- Let x(n) be a discrete signal. Its ZT is defined by: $\boxed{X(z)=\sum_{n=-\infty}^{+\infty} x(n).z^{-n}}$

where z is a complex variable defined wherever this series converges.

A region of convergence corresponds to the set of values of z such that X(z) is defined and has finite values.

Examples

1. $x(n) = \delta(n) \Rightarrow X(z) = 1$ ROC=¢

2. $x(n) = \delta(n-k) \Rightarrow X(z) = z^{-k}$ If k>0 ROC=¢-{0} if k<0 ROC=¢-{ ∞ }

3. $x(n)=(\underline{1},2,3,5,0,2) \Rightarrow x(n)= \delta(n)+2.\delta(n-1)+3.\delta(n-2)+5.\delta(n-3)+2.\delta(n-5)$

$\Rightarrow$ $$X(z)=1+2z^{-1}+3z^{-2}+5z^{-3}+2z^{-5}$$

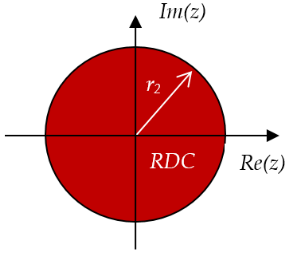
II. Z Transform

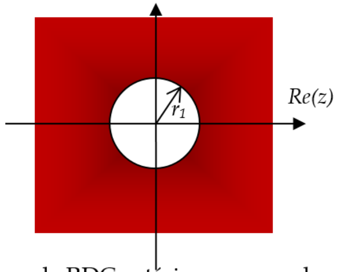
Examples

$$x(n) = \begin{cases} a^n \ \ si\ n \geq 0 \\ 0\ si\ non \end{cases} \qquad X(z) = \sum_{n=0}^{+\infty} a^n z^{-n} = \frac{z}{z-a} \qquad |z| > a$$

$$y(n) = \begin{cases} 0\ si\ n \geq 0 \\ -b^n \ \ si\ n < 0 \end{cases} \qquad Y(z) = \sum_{n<0}^{-\infty} -b^n z^{-n} = \frac{z}{z-b} \qquad |z| < b$$

$$w(z) = \begin{cases} a^n \ \ si\ n \geq 0 \\ b^n \ \ si\ n < 0 \end{cases} \qquad W(z) = \frac{z}{z-a} - \frac{z}{z-b} \qquad b > |z| > a$$





Note: The TZ of a^n for n $\in$]- ∞ , + ∞ [does not exist.

$\Rightarrow$ système anti-causal : RDC cercle   système causal : RDC extérieure au cercle.

II. Z Transform: Rational ZT

Invariant linear systems described by a finite difference equation have a rational Z-transform (the ratio of two polynomials in z $^{-1}$)

$$\sum_{i=0}^{M} a_i y(n-i) = \sum_{i=0}^{N} b_i x(n-i) \xrightarrow{TZ} \sum_{i=0}^{M} a_i . z^{-i} Y(z) = \sum_{i=0}^{N} b_i . z^{-i} X(z)$$

A LI system can be characterized by h(n) or by the Z transform (H(z)) called the system transfer function. of its impulse response h(n), called the system transfer function.

$$H(z) = \frac{\sum_{i=0}^{N} b_i \, .z^{-i}}{\sum_{i=0}^{M} a_i \, .z^{-i}} = \frac{b_0}{a_0} z^{M-N} \frac{N(z)}{D(z)} = \frac{b_0}{a_0} z^{M-N} \frac{\prod_{i=1}^{N} (z - z_i)}{\prod_{i=1}^{M} (z - p_i)} = K \, z^{M-N} \frac{\prod_{i=1}^{N} (z - z_i)}{\prod_{i=1}^{M} (z - p_i)}$$

The values of z for which H(z)=0 are called zeros

The values of z for which H(z) is infinite (zeroes the denominator) are called the poles

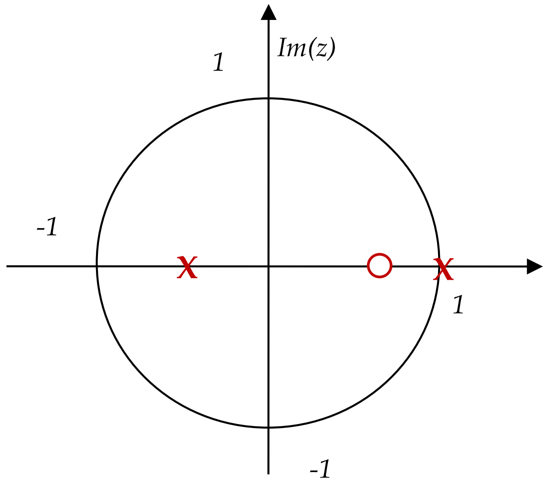
II. Z Transform: Representation of poles and zeros

The position of its poles and zeros (+ the amplitude factor $K = b_0 / a_0$) will provide us with a complete description of H(z) (consequently of h(n) and H(f)) and therefore of the behavior of the system.

H(z) can therefore be represented as a circle modeling the position of the poles and zeros in the complex plane.

Example

$$H(z) = \frac{3z-2}{(z-1)(z+0.5)}$$

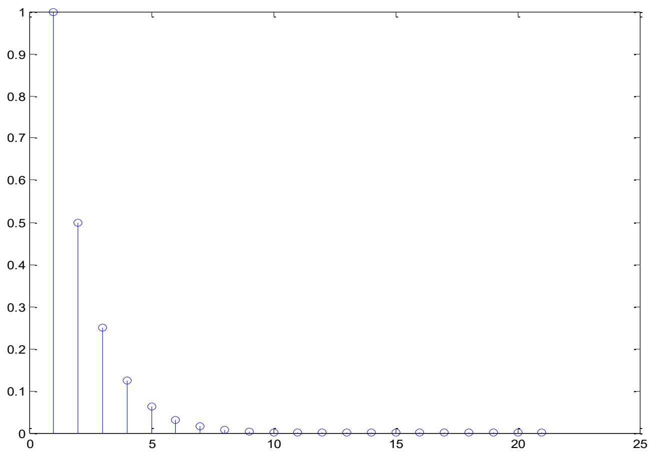


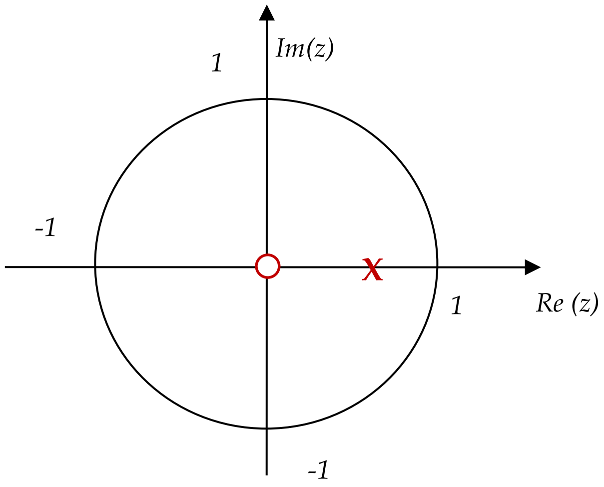
II. Z Transform: Representation of poles and zeros

Determination of the impulse response from the pole-zero plot

$$h(n) = a^n\, U(n) \Rightarrow H(z) = \frac{z}{z-a}$$

- For 0 < a < 1





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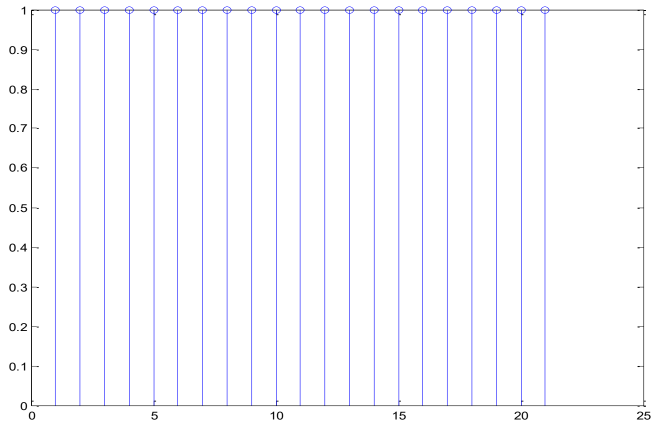

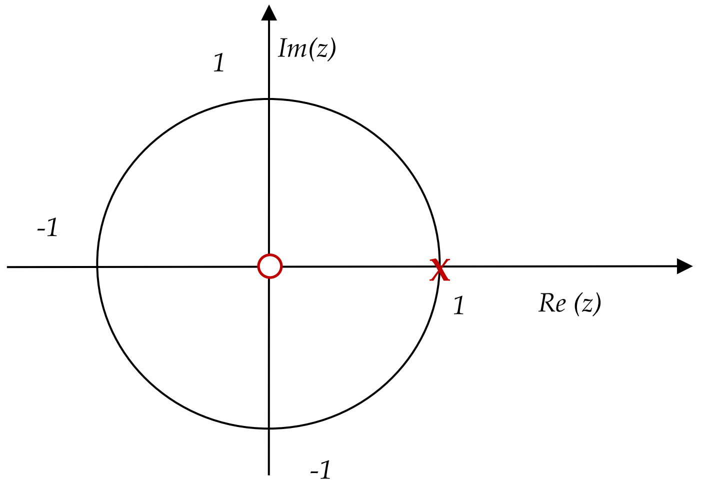

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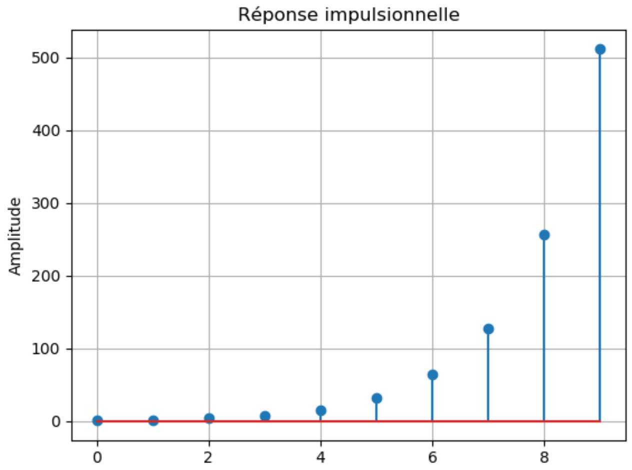

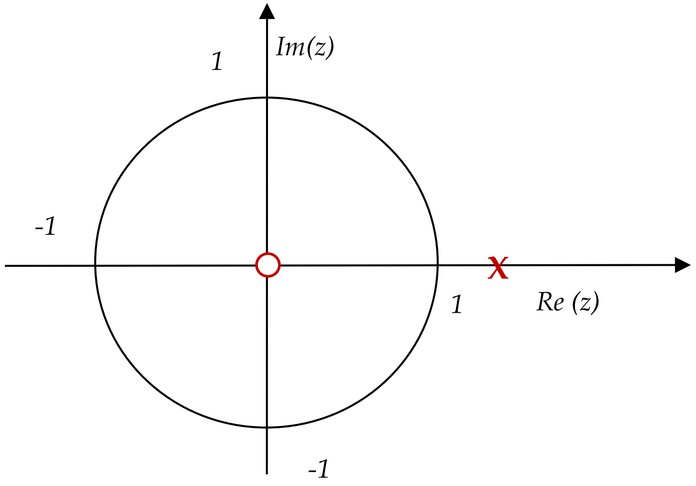

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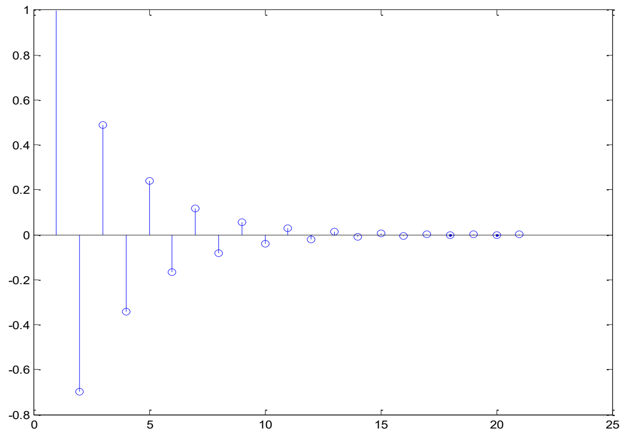
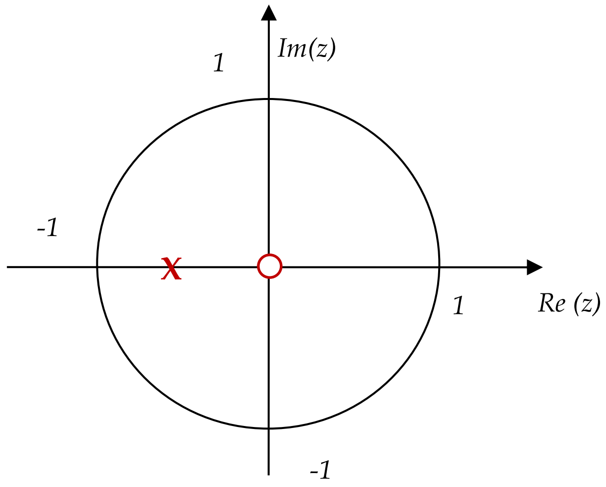
- For -1 <a < 0





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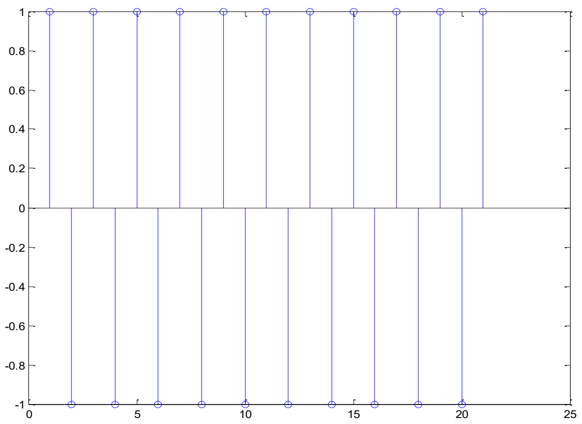

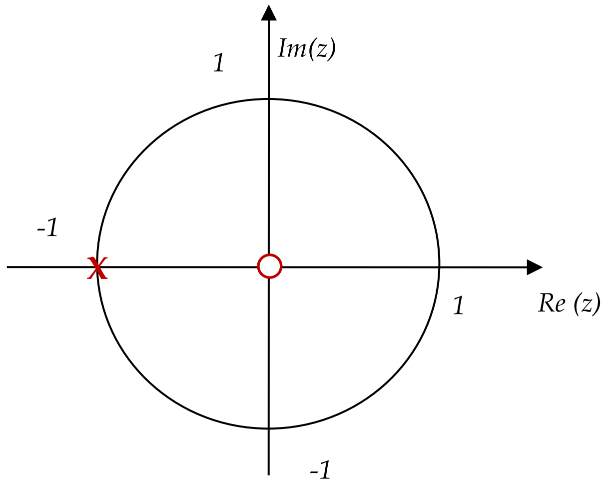

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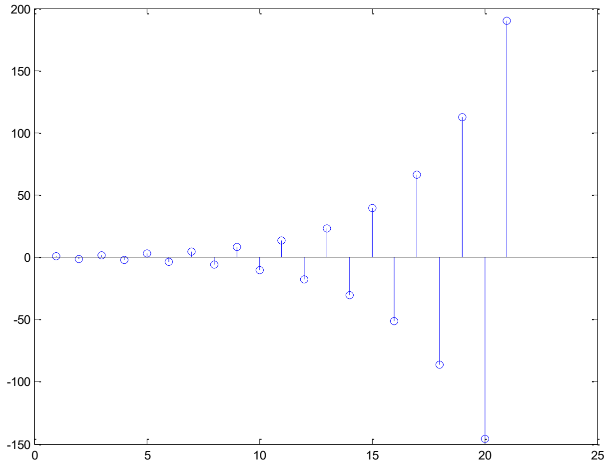
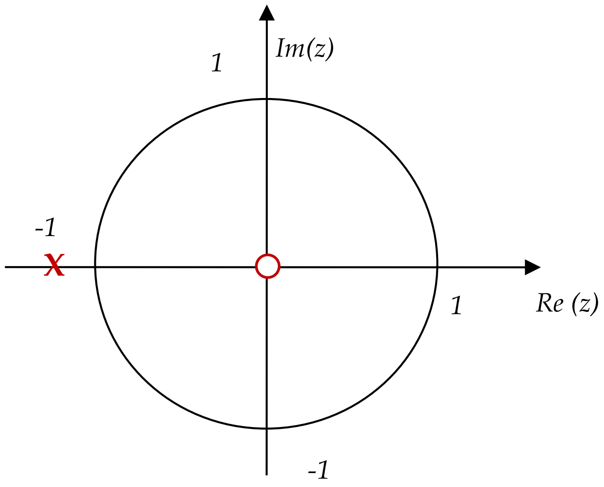


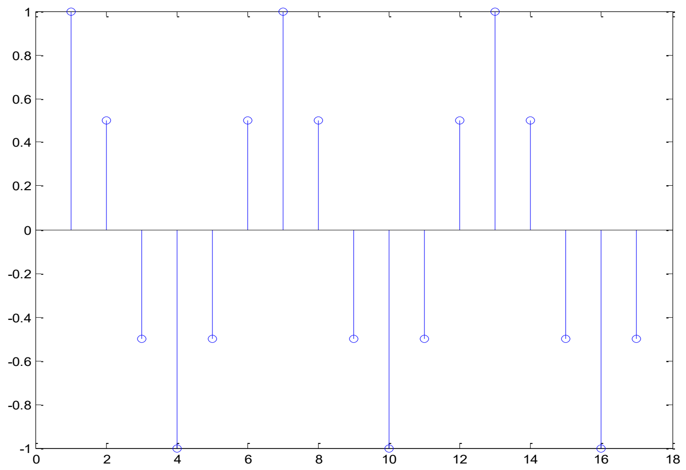


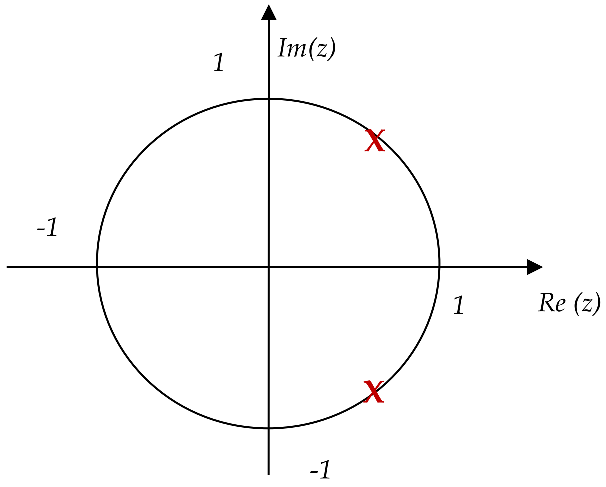
II. Z Transform: Representation of poles and zeros

Determination of the impulse response from the pole-zero plot

$$h(n) = a^n \cos(2\pi f_0 n)\, U(n) \quad => \quad \mathrm{H(z)} = \frac{\dots\dots\dots..}{(z - ae^{2\pi j f_0 n})(z - ae^{-2\pi j f_0 n})}$$

- For a = 1





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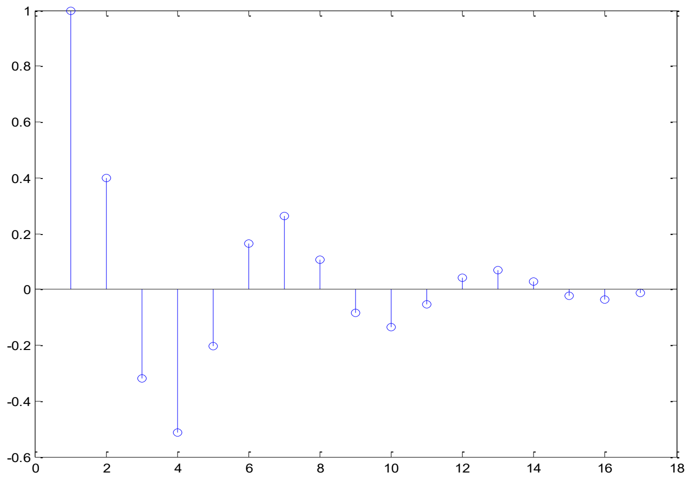

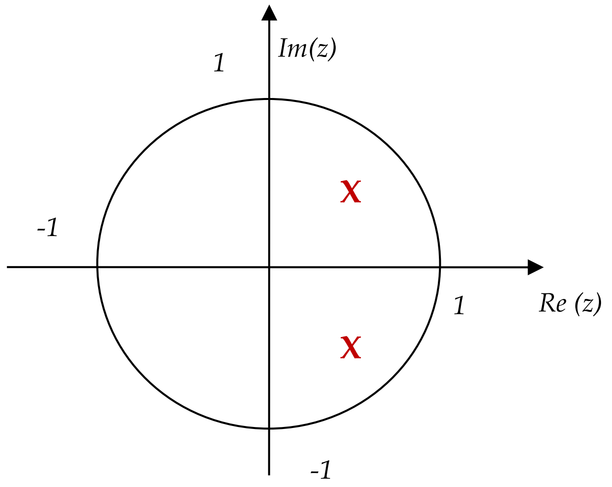

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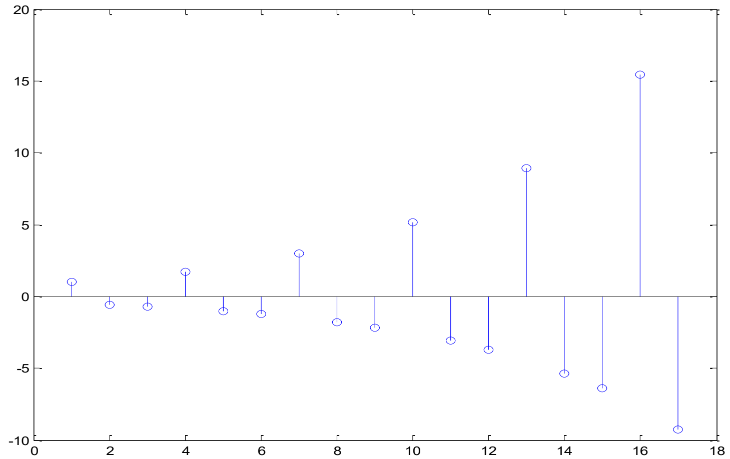
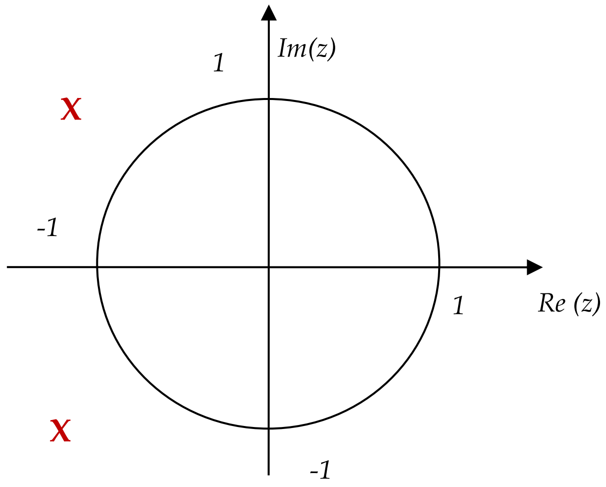
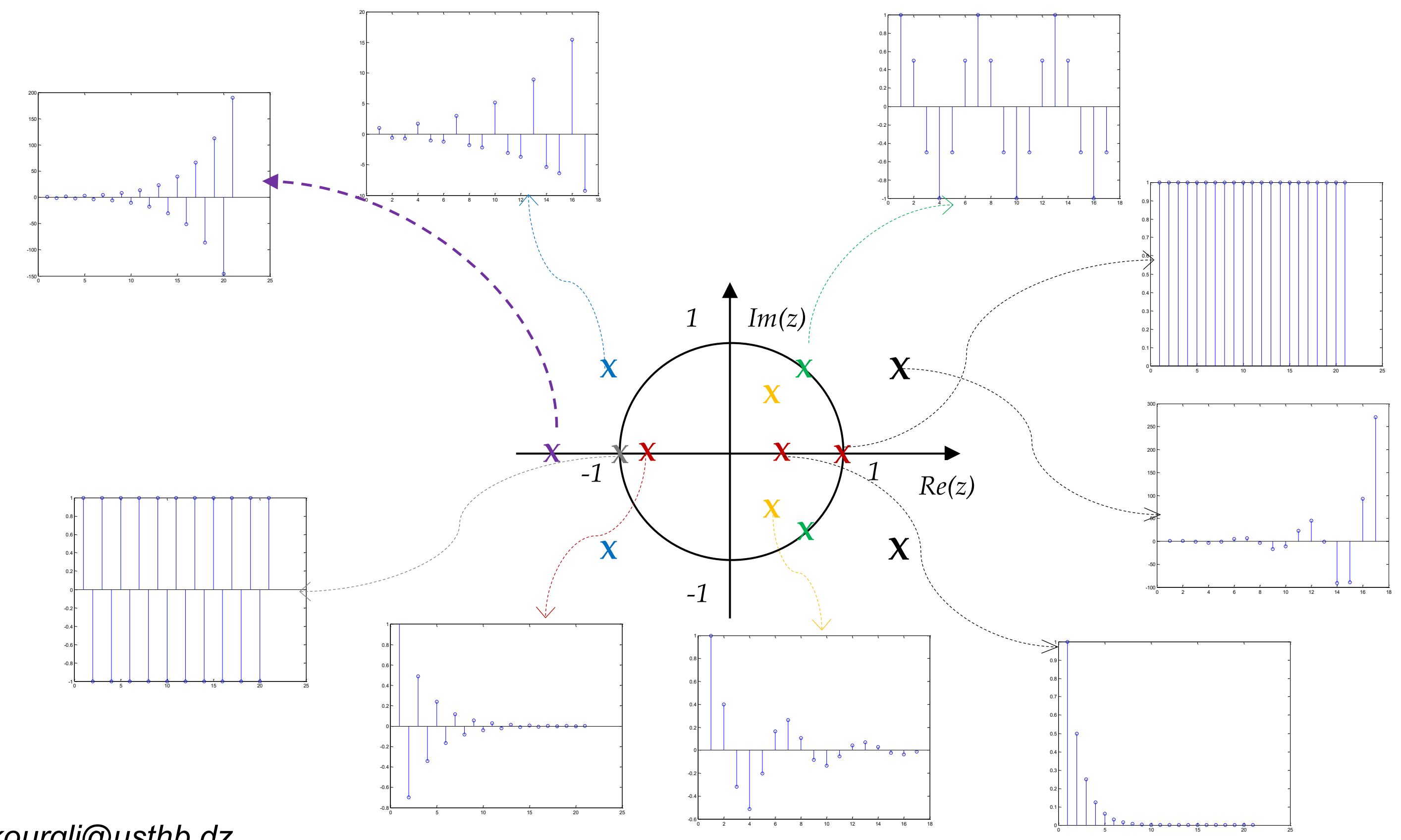




II. Z Transform: Representation of poles and zeros

Determination of the impulse response from the pole-zero plot



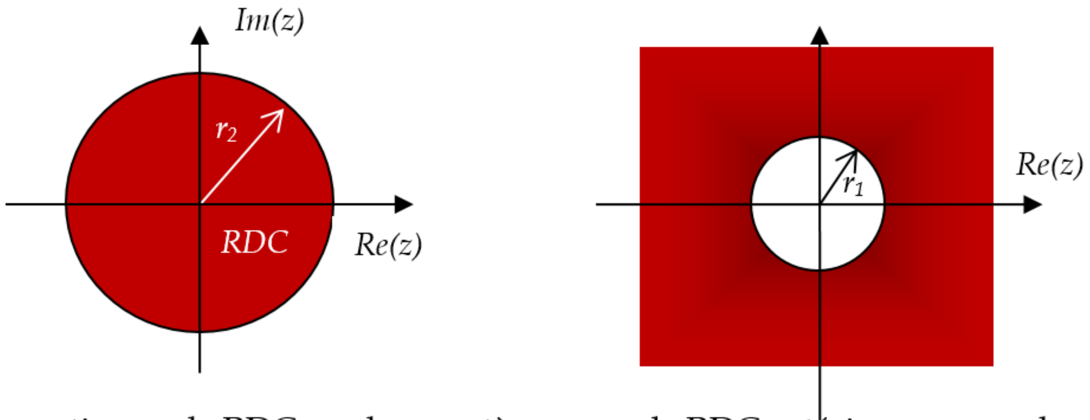
II. Z Transform: Representation of poles and zeros

✓In most systems, ai and bi are real $\Rightarrow$ poles and zeros are either real or complex conjugate pairs.

✓The radius of a causal system lies outside a circle. Moreover, if it is stable $\sum_n |h(n)| < \infty$

since $H(z) = \sum_{n=-\infty}^{+\infty} h(n).z^{-n}$

$\Rightarrow$ it is therefore sufficient that z=1 is part of the DRC



$\Rightarrow$ système anti-causal : RDC cercle système causal : RDC extérieure au cercle.

II. Z Transform: Representation of poles and zeros

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$\Rightarrow$ it is therefore sufficient that z=1 is part of the ROC

- ✓ For a causal and stable system, all poles are inside the unit circle (|pi|<1, $\forall$ i). The domain of convergence cannot contain poles since the TZ does not converge at the poles. If it is anti-causal, it will be stable if the poles are outside the unit circle.
- ✓A FIR filter has all its poles at the origin and will therefore always be stable.

$$H(z) = \sum_{i=0}^{N} b_i .z^{-i}$$

II. Z Transform: Representation of poles and zeros

Determination of the frequency response from the pole-zero plot

- For a SLID

$$H(z) = K\, z^{M-N} \frac{\prod_{i=1}^{N} (z - z_i)}{\prod_{i=1}^{M} (z - p_i)}$$

- Transition from H(z) to H(f)

$$H(z) = \sum_{n=-\infty}^{+\infty} h(n).z^{-n}$$

$$H(f) = \sum_{n=-\infty}^{+\infty} h(nT_e) e^{-2\pi jf\, nTe}$$

$$H(f) = H(z)\Big|_{z=e^{2\pi jfTe}}$$

II. Z Transform: Representation of poles and zeros

Determination of the frequency response from the pole-zero plot

$$H(f) = \sum_{n=-\infty}^{+\infty} h(nT_e)e^{-2\pi jf\,nTe} = H(z)\Big|_{z=e^{2\pi jfTe}}$$

$$\Rightarrow \quad |z| = \left|e^{2\pi jfTe}\right| = \left|\cos(2\pi fTe) + j\sin(2\pi fTe)\right| = 1$$

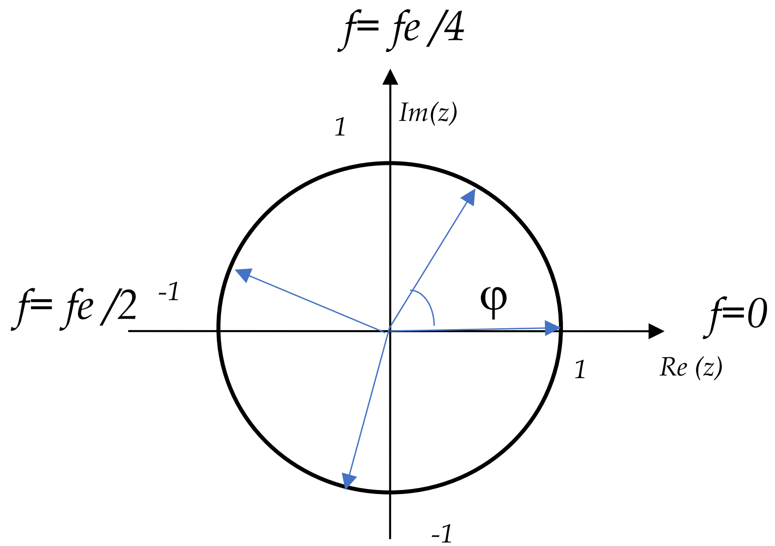
$$\Rightarrow \quad z = e^{2\pi jfTe} = e^{j\varphi} \;\; où \;\; \varphi = 2\pi fTe$$

$$\rightarrow f = 0 \; où \;\; \varphi = 2\pi 0Te = 0$$

$$\rightarrow f = fe/4 \, où \;\; \varphi = 2\pi fe/4Te = \pi/2$$

$$\rightarrow f = fe/2 \, où \;\; \varphi = 2\pi fe/4Te = \pi$$

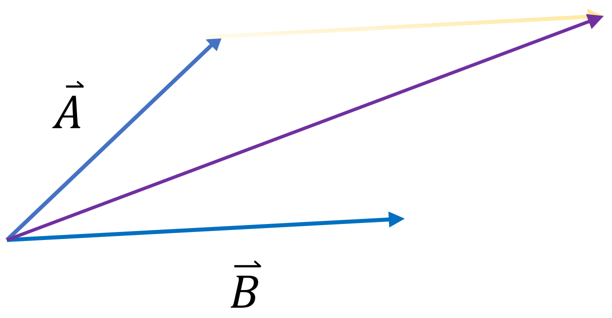
$$\rightarrow f = fe \; où \;\; \varphi = 2\pi feTe = 2\pi$$



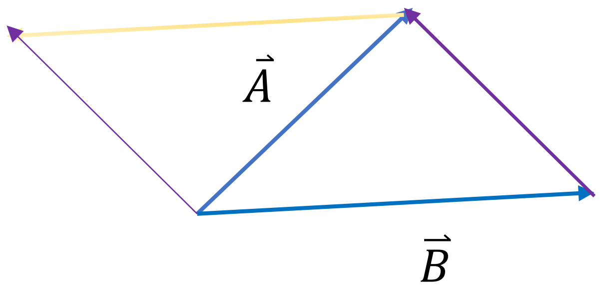
II. Z Transform: Representation of poles and zeros

Determination of the frequency response from the pole-zero plot

- Reminder on vectors
- Sum



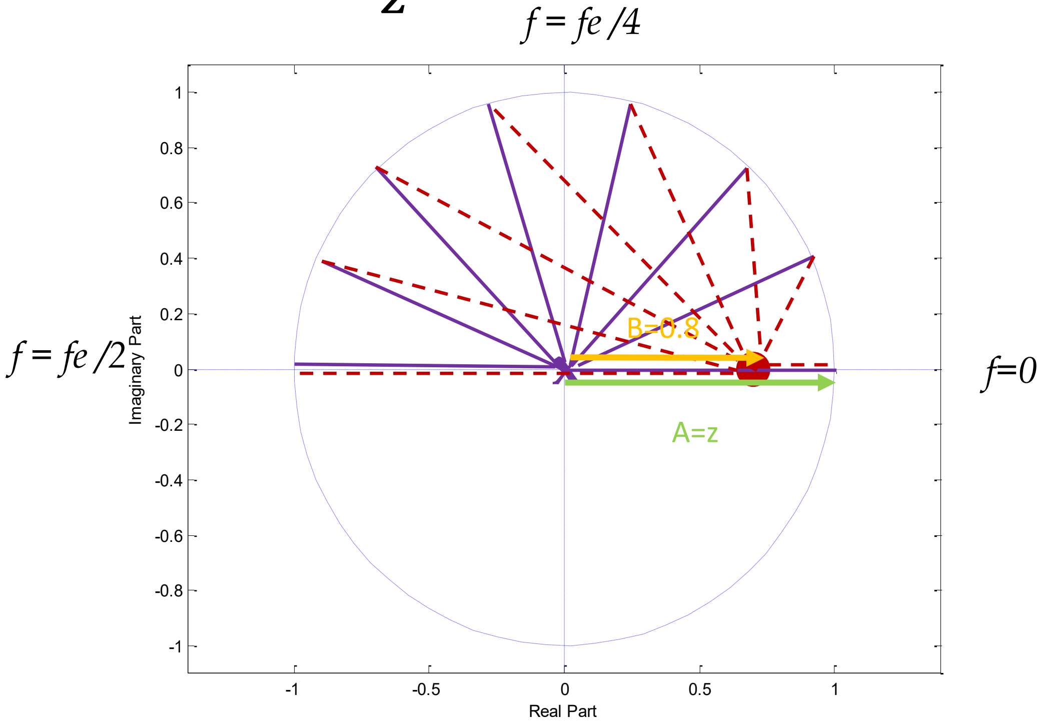
- Difference



Determination of the frequency response from the pole-zero plot

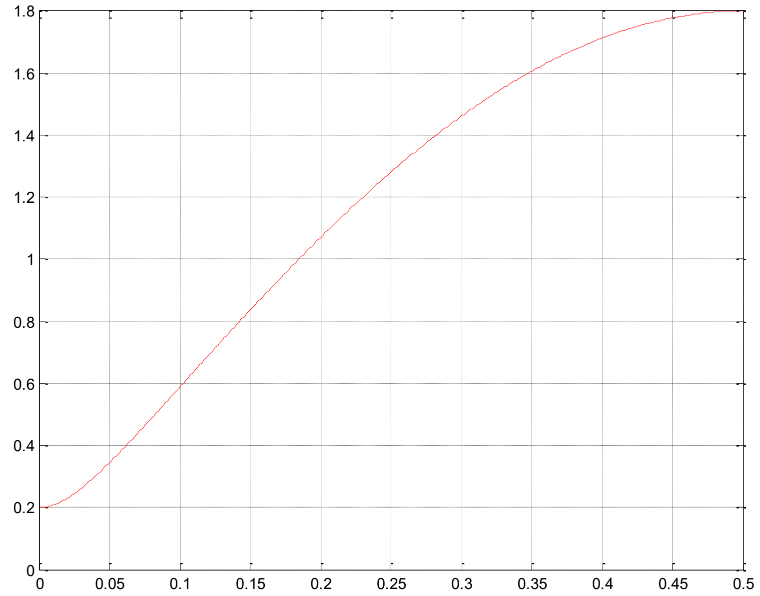
$$H(f) = \sum_{n=-\infty}^{+\infty} h(nT_e) e^{-2\pi jfnTe} = H(z)\Big|_{z=e^{2\pi jfTe}} = K\, z^{M-N} \left. \frac{\prod_{i=1}^{N}(z-z_i)}{\prod_{i=1}^{M}(z-p_i)} \right|_{z=e^{2\pi jfTe}}$$

Example 1: $H(z) = \dfrac{(z-0.8)}{z}$



$\Rightarrow$

$|H(f)|$



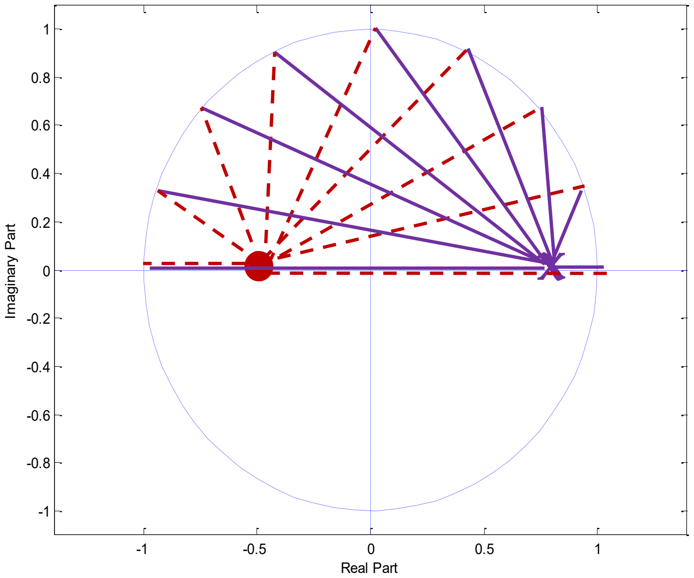
II. Z Transform: Representation of poles and zeros

Determination of the frequency response from the pole-zero plot

Example 2 : $H(z) = \frac{(z+0.5)}{(z-0.8)}$

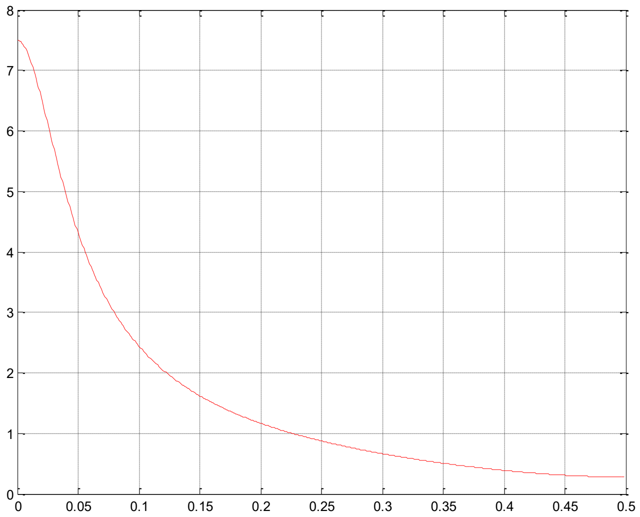
$f = fe/4$

$f = fe/2$



$f=0$ $\Rightarrow$

$|H(f)|$

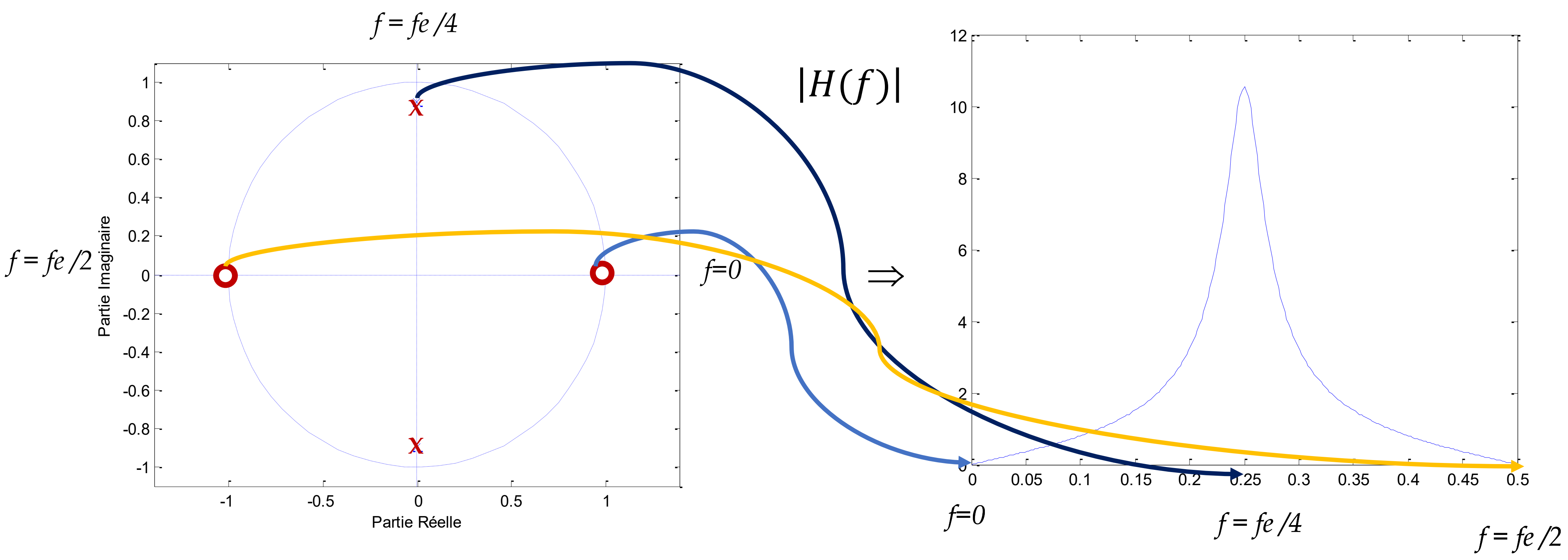


- A zero or a pole at the origin does not influence the modulus of the frequency response.
- A zero near the unit circle introduces an attenuation in the frequency response modulus (minimum)
- A pole in the vicinity of the unit circle introduces a resonance that is all the more important in the modulus of the frequency response as the pole is close to the unit circle (maximum)

II. Z Transform: Representation of poles and zeros

Determination of the frequency response from the pole-zero plot

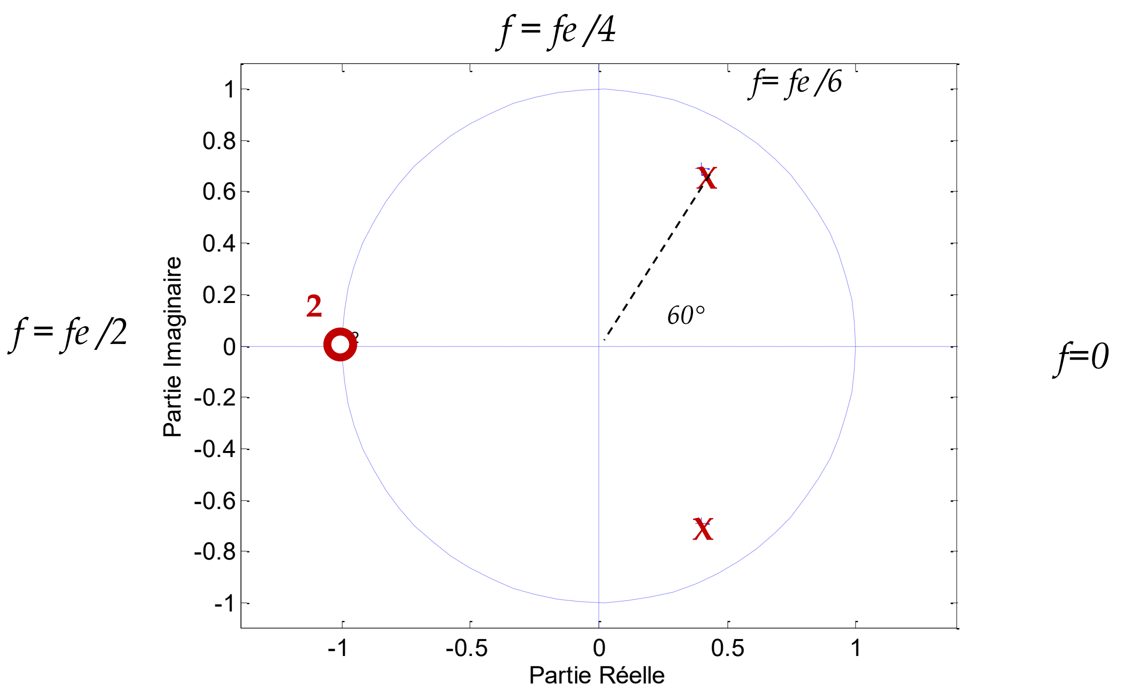
Example 3



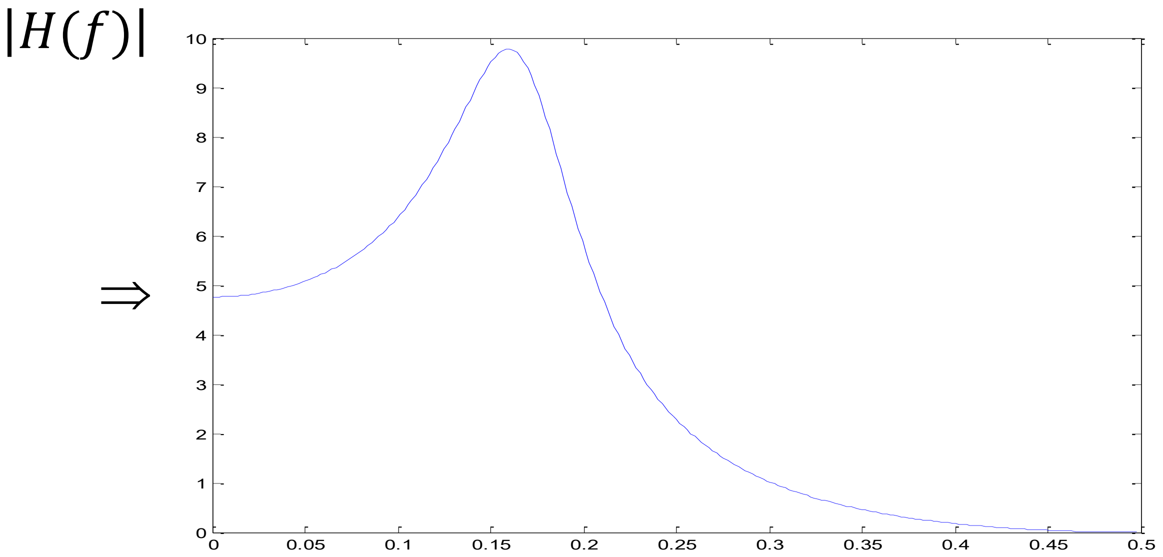
II. Z Transform: Representation of poles and zeros

Determination of the frequency response from the pole-zero plot

Example 4



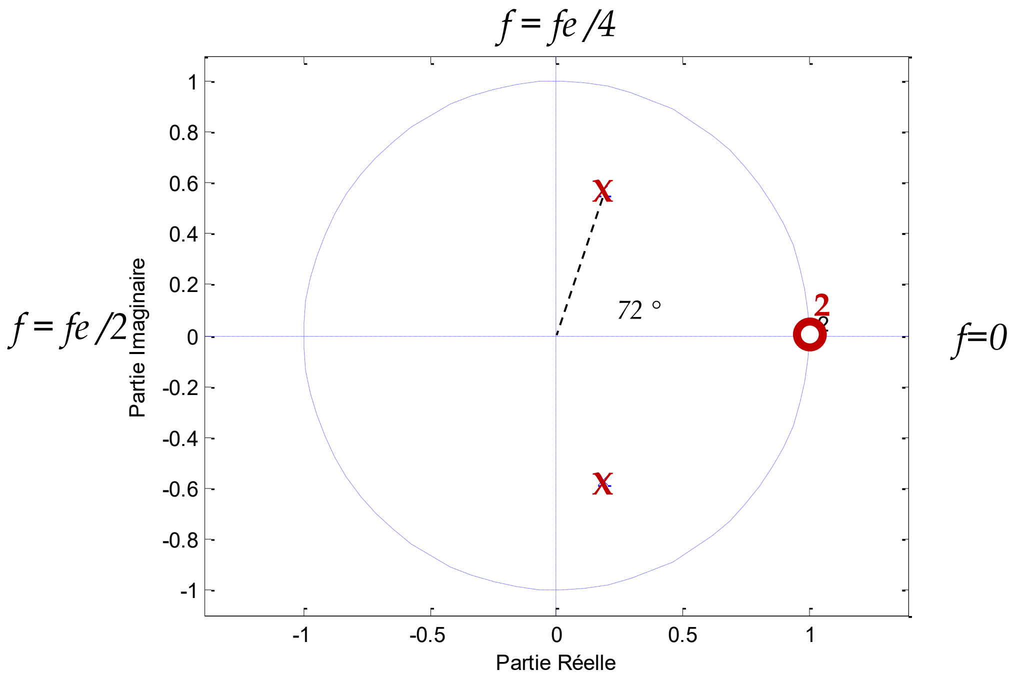
$\Rightarrow$



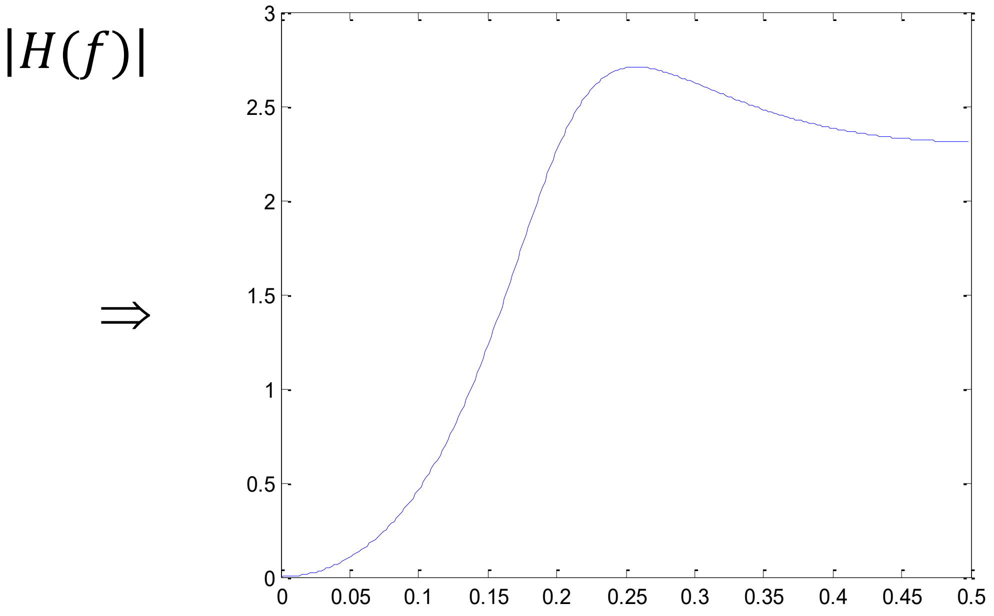
II. Z Transform: Representation of poles and zeros

Determination of the frequency response from the pole-zero plot

Example 5



$\Rightarrow$



II. Z Transform: TZ Inverse

Residue Method

$$h(n) = \sum_{p_i \text{ poles de } z^{n-1}H(z)} \operatorname{Re} s\left[z^{n-1}H(z)\right]_{z=pi}$$

$$\operatorname{Re} s[z^{n-1}H(z)]_{z=pi} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-p_i)^m z^{n-1}H(z)\right]_{z=pi}$$

Example

$\Rightarrow$ 1 single pole e^{-a}(m=1)

$$H(z) = \frac{z}{z-e^{-a}}$$

$$\operatorname{Re} s[z^{n-1}H(z)]_{z=e^{-a}} = \left[(z-e^{-a})z^{n-1}\frac{z}{z-e^{-a}}\right]_{z=e^{-a}} = \left[z^n\right]_{z=e^{-a}} = e^{-an}.u(n)$$

II. Z Transform: TZ Inverse

Development in simple Z functions

$x(n)$	$X(z)$	Convergence region
$\delta(n)$	1	$\forall z$
$U(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$a^n U(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$n a^n U(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-a^n U(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$\cos(\omega_0 n T_e) U(n)$	$\frac{1-z^{-1}\cos(\omega_0 T_e)}{1-2z^{-1}\cos(\omega_0 T_e)+z^{-2}}$	$\|z\| > 1$
$\sin(\omega_0 n T_e) U(n)$	$\frac{z^{-1}\sin(\omega_0 T_e)}{1-2z^{-1}\cos(\omega_0 T_e)+z^{-2}}$	$\|z\| > 1$
$a^n \cos(\omega_0 n T_e) U(n)$	$\frac{1-az^{-1}\cos(\omega_0 T_e)}{1-2az^{-1}\cos(\omega_0 T_e)+a^2 z^{-2}}$	$\|z\| > \|a\|$
$a^n \sin(\omega_0 n T_e) U(n)$	$\frac{az^{-1}\sin(\omega_0 T_e)}{1-2az^{-1}\cos(\omega_0 T_e)+a^2 z^{-2}}$	$\|z\| > \|a\|$

II. Z Transform: TZ Inverse

Inverse Transform by Polynomial Division

Example

$$H(z)=\frac{1}{1-a.z^{-1}}=\frac{z}{z-a} \quad pour \ |z|>|a|$$

Domain of convergence outside a circle → causal signal → division to have a series in z $^{-1}$

$1 \quad 1-az^{-1}$

$-(1-az^{-1})$

az^{-1}

$-(az^{-1}-a^2z^{-2})$

a^2z^{-2}

$1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \ldots\ldots$

h(0) h(1) h(2) h(3)

$$\Rightarrow \qquad h(n)=a^n.U(n)$$

II. Z Transform: TZ Inverse

Inverse Transform by Polynomial Division

Example

$$H(z)=\frac{1}{1-a.z^{-1}}=\frac{z}{z-a} \quad pour \ |z|<|a|$$

Domain of convergence inside a circle → anti-causal signal → division to have a z-series

z - a+z

- $(za^{-1}z^{2})$

$a^{-1}z^{2}$

$-(a^{-1}z^{2}-a^{-2}z^{3})$

$a^{-2}z^{3}$

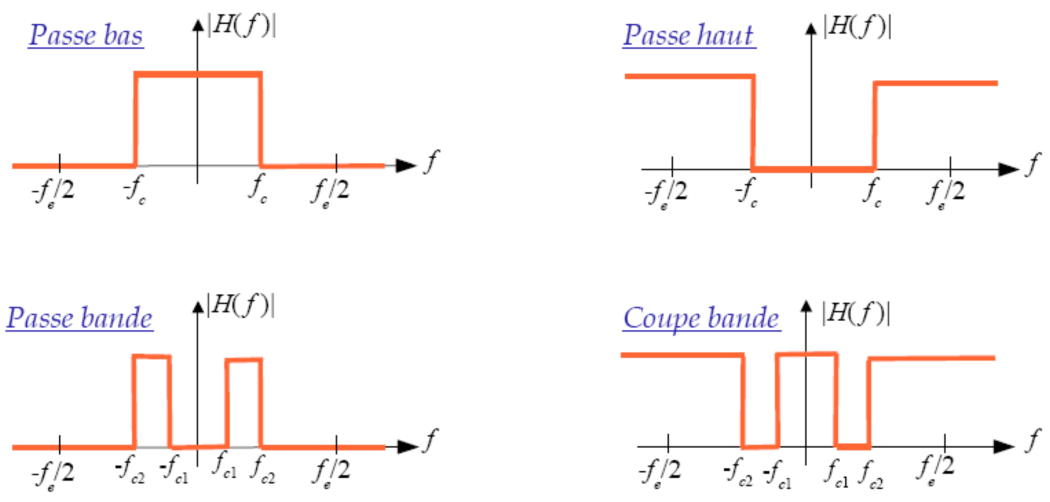
$-a^{-1}z \quad -a^{-2}z^{2} \quad -a^{-3}z^{3} \quad - \ldots\ldots.$

h(-1) h(-2) h(-3)

$$\Rightarrow \qquad h(n)=-a^{n}.u(-n-1)$$

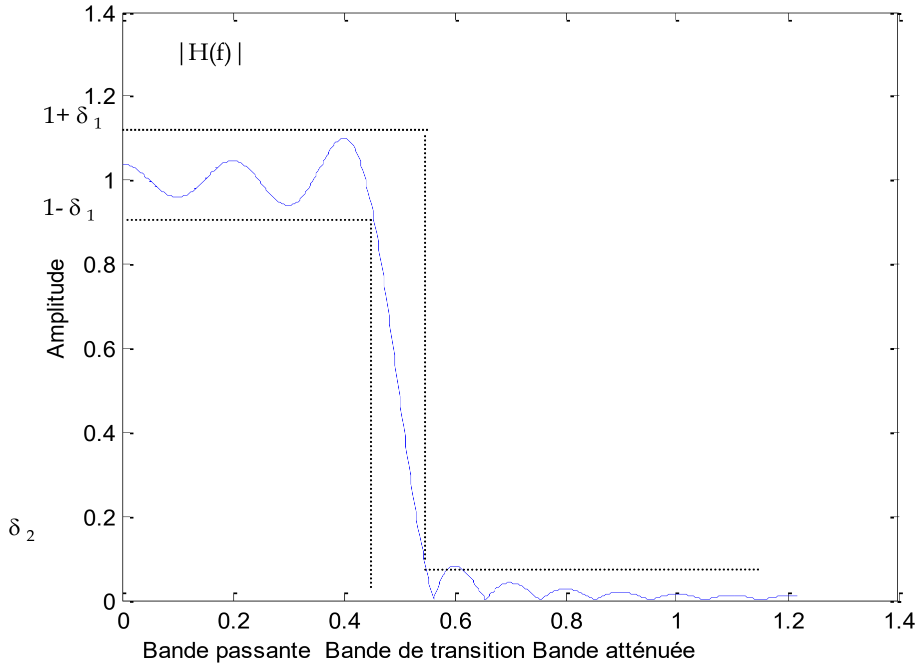
III. Characteristics of Digital Filters

- Ideal filter



- FIR or IIR?

- Real filter



III. Characteristics of Digital Filters

- FIR or IIR?

$$\sum_{i=0}^{M} a_i y(n-i) = \sum_{i=0}^{N} b_i x(n-i) \xrightarrow{TZ} \sum_{i=0}^{M} a_i . z^{-i} Y(z) = \sum_{i=0}^{N} b_i . z^{-i} X(z) \Longrightarrow H(z) = \frac{\sum_{i=0}^{N} b_i . z^{-i}}{\sum_{i=0}^{M} a_i . z^{-i}} = K\, z^{M-N} \frac{\prod_{i=1}^{N} (z - z_i)}{\prod_{i=1}^{M} (z - p_i)}$$

Reminders

- IIR: $\sum_{i=0}^{M} a_i y(n-i) = \sum_{i=0}^{N} b_i x(n-i)$

- FIR: $y(n) = \sum_{i=0}^{N} b_i x(n-i) \quad \Rightarrow \quad H(z) = \sum_{i=0}^{N} b_i z^{-i} = \frac{b_0 \prod_{i=1}^{N} (z - z_i)}{z^N} \quad \Rightarrow \quad h(n) = \sum_{i=0}^{N} b_i \delta(n-i)$

All poles are zeros $\Rightarrow$ FIR always stable

+ Possibility to have a constant group delay

III. Characteristics of Digital Filters

- **Group delay**

$$H(f) = |H(f)|e^{-j\phi(f)} \qquad \beta = -\frac{d\phi(f)}{df}$$

- Reminder $TF\{x(t-t_0)\} = X(f)e^{-2\pi jft_0}$

- A linear phase will ensure the same phase shift for all frequencies (no distortion).

$$\phi(f) = \phi_0 + 2\pi\, f\, t_0 \qquad \beta = -\frac{d\phi(f)}{df} = 2\pi t_0 = cste$$

- FIR filters can generate linear phase filters provided that the impulse response is symmetrical

III. Characteristics of Digital Filters

Example

$$y(n) = \frac{1}{4}\left(x(n) + 2x(n-1) + x(n-2)\right)$$

$$\Rightarrow \quad h(n) = \frac{1}{4}\left(\delta(n) + 2\delta(n-1) + \delta(n-2)\right)$$

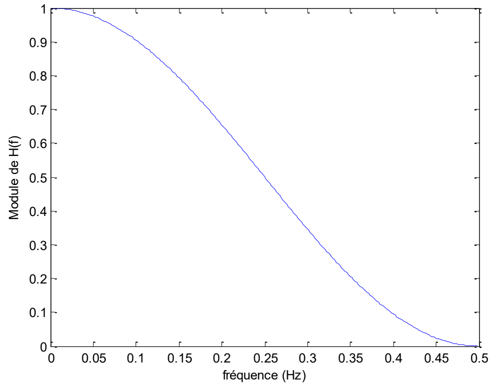
$$\Rightarrow \quad H(z) = \frac{1}{4}(1 + 2z^{-1} + z^{-2})$$

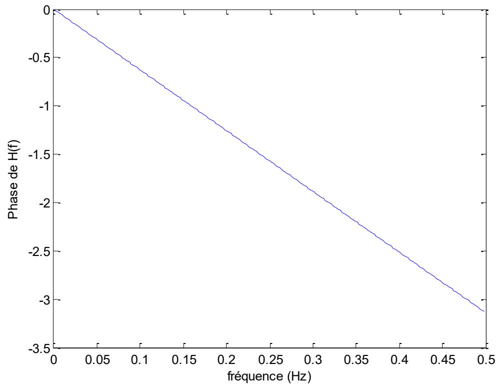
$$\Rightarrow \quad H(f) = H(z)\Big|_{z=e^{2\pi j f T_e}} = \frac{1}{4}(1 + 2e^{-2\pi j f T_e} + e^{-4\pi j f T_e})$$

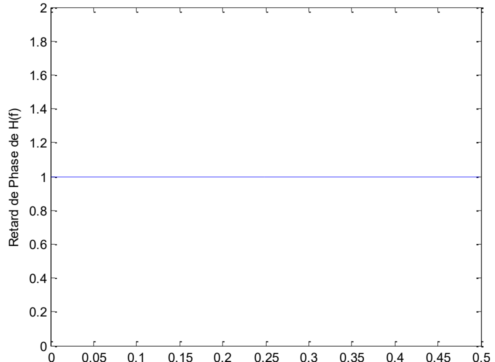
$$H(f) = \frac{1}{4}e^{-2\pi j f T_e}\,(e^{2\pi j f T_e} + 2 + e^{-2\pi j f T_e})$$

$$H(f) = \frac{1}{4}e^{-2\pi j f T_e}\,(2\cos(2\pi f T_e) + 2) = e^{-2\pi j f T_e}\,(cos^2(\pi f T_e))$$

$$\Rightarrow \quad \beta = -\frac{d\varphi(f)}{df} = 2\pi T_e = cste$$







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III. Characteristics of Digital Filters

FIR or IIR?

- IIR

➢Desired frequency response with very few poles and zeros

☻Risk of instability (recursion)

- FIR

✓Still stable

✓Possibility of having a constant group delay

☻No poles outside zeros (long impulse response)