CONVEX FOLIATIONS OF DEGREE 4 ON THE COMPLEX PROJECTIVE PLANE

by

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Abstract. — We show that up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$ there are 5 homogeneous convex foliations of degree four on $\mathbb{P}^2_{\mathbb{C}}$. Using this result, we give a partial answer to a question posed in 2013 by D. MARÍN and J. PEREIRA about the classification of reduced convex foliations on $\mathbb{P}^2_{\mathbb{C}}$.
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Introduction

The set $\mathbf{F}(d)$ of foliations of degree d on $\mathbb{P}^2_{\mathbb{C}}$ can be identified with a ZARISKI open subset of the projective space $\mathbb{P}^{(d+2)^2-2}_{\mathbb{C}}$. The group of automorphisms of $\mathbb{P}^2_{\mathbb{C}}$ acts on $\mathbf{F}(d)$. The orbit of an element $\mathcal{F}\in\mathbf{F}(d)$ under the action of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})=\mathrm{PGL}_3(\mathbb{C})$ will be denoted by $\mathcal{O}(\mathcal{F})$. Following [10] we will say that a foliation in $\mathbf{F}(d)$ is *convex* if its leaves other than straight lines have no inflection points. The subset $\mathbf{FC}(d)$ of $\mathbf{F}(d)$ consisting of all convex foliations is ZARISKI closed in $\mathbf{F}(d)$.
By [3, Proposition 2, page 23] every foliation of degree 0 or 1 is convex, *i.e.* $\mathbf{FC}(0)=\mathbf{F}(0)$ and $\mathbf{FC}(1)=\mathbf{F}(1)$. For $d\geq 2$, $\mathbf{FC}(d)$ is a proper closed subset of $\mathbf{F}(d)$ and it contains the FERMAT foliation $\mathcal{F}_0^d$ of degree d, defined in the affine chart (x,y) by the 1-form (*see* [10, page 179])

$$\overline{\omega}_0^d=(x^d-x)\mathrm{d}y-(y^d-y)\mathrm{d}x.$$

The closure inside $\mathbf{F}(d)$ of the orbit of $\mathcal{F}_0^d$ contains the foliations $\mathcal{H}_0^d$, resp. $\mathcal{H}_1^d$, resp. $\mathcal{F}_1^d$ (necessarily convex) defined by the 1-forms (*see* [4, Example 6.5] and [5, page 75])

$$\omega_0^d=(d-1)y^d\mathrm{d}x+x(x^{d-1}-dy^{d-1})\mathrm{d}y,\quad \text{resp. } \omega_1^d=y^d\mathrm{d}x-x^d\mathrm{d}y,\quad \text{resp. } \overline{\omega}_1^d=y^d\mathrm{d}x+x^d(x\mathrm{d}y-y\mathrm{d}x).$$

In other words, we have the following inclusions

$$\mathcal{O}(\mathcal{H}_0^d)\cup\mathcal{O}(\mathcal{H}_1^d)\cup\mathcal{O}(\mathcal{F}_0^d)\cup\mathcal{O}(\mathcal{F}_1^d)\subset\overline{\mathcal{O}(\mathcal{F}_0^d)}\subset\mathbf{FC}(d). \tag{0.1}$$

Key words and phrases. — convex foliation, homogeneous foliation, singularity, inflection divisor.

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The foliations $\mathcal{H}_0^d$ and $\mathcal{H}_1^d$ are *homogeneous*, *i.e.* they are invariant by homotheties; moreover, they are linearly conjugated for $d=2$, but not for $d\geq 3$, *see* [4]. The dimension of the orbit of $\mathcal{F}_1^d$ is 6 [5], which is the least possible dimension in any degree d greater or equal to 2 ([7, Proposition 2.3]). Notice (*see* [5]) that this bound is also attained by the non convex foliation $\mathcal{F}_2^d$ defined by the 1-form

$$\overline{\omega}_2^d = x^d\mathrm{d}x + y^d(x\mathrm{d}y - y\mathrm{d}x).$$

The classification of the elements of $\mathbf{FC}(2)$ has been established by C. FAVRE and J. PEREIRA [9, Proposition 7.4]: up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$, the foliations $\mathcal{H}_0^2$, $\mathcal{F}_0^2$ and $\mathcal{F}_1^2$ are the only convex foliations of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$. This classification implies that in degree 2 the inclusions (0.1) are equalities:

$$\mathbf{FC}(2) = \overline{\mathcal{O}(\mathcal{F}_0^2)} = \mathcal{O}(\mathcal{H}_0^2) \cup \mathcal{O}(\mathcal{F}_0^2) \cup \mathcal{O}(\mathcal{F}_1^2). \tag{0.2}$$

For dimensional reasons the orbits $\mathcal{O}(\mathcal{F}_1^d)$ and $\mathcal{O}(\mathcal{F}_2^d)$ are closed; combining equalities (0.2) with [7, Theorem 3], we see, in particular, that the only closed orbits in $\mathbf{F}(2)$ by the action of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ are those of $\mathcal{F}_1^2$ and $\mathcal{F}_2^2$.
Convex foliations of degree 3 has been classified by the first author in his thesis [5, Corollary C]: every foliation $\mathcal{F} \in \mathbf{FC}(3)$ is linearly conjugated to one of the four foliations $\mathcal{H}_0^3$, $\mathcal{H}_1^3$, $\mathcal{F}_0^3$ or $\mathcal{F}_1^3$. This implies that the inclusions (0.1) for $d=3$ are also equalities:

$$\mathbf{FC}(3) = \overline{\mathcal{O}(\mathcal{F}_0^3)} = \mathcal{O}(\mathcal{H}_0^3) \cup \mathcal{O}(\mathcal{H}_1^3) \cup \mathcal{O}(\mathcal{F}_0^3) \cup \mathcal{O}(\mathcal{F}_1^3). \tag{0.3}$$

For $d\geq 4$, the classification of the elements of $\mathbf{FC}(d)$ modulo $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ remains open and the topological structure of $\mathbf{FC}(d)$ is not yet well understood. In the sequel we will focus on the case $d=4$. Notice (*see* [10, page 181]) that the set $\mathbf{FC}(4)$ contains the foliation $\mathcal{F}_H^4$, called HESSE pencil of degree 4, defined by

$$\omega_H^4 = (2x^3 - y^3 - 1)y\mathrm{d}x + (2y^3 - x^3 - 1)x\mathrm{d}y\ ;$$

furthermore $\mathcal{O}(\mathcal{F}_H^4) \neq \mathcal{O}(\mathcal{F}_0^4)$ and $\dim \mathcal{O}(\mathcal{F}_H^4) = \dim \mathcal{O}(\mathcal{F}_0^4) = 8$. So that the inclusion $\overline{\mathcal{O}(\mathcal{F}_0^4)} \subset \mathbf{FC}(4)$ is strict, in contrast to the previous cases of degrees 2 and 3.
In this paper we propose to classify, up to automorphism, the foliations of $\mathbf{FC}(4)$ which are homogeneous, *i.e.* which are invariant under the $\mathbb{C}^*-$action $(x,y)\mapsto(tx,ty)$. More precisely, we establish the following theorem.

Theorem A. — *Up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$ there are five homogeneous convex foliations of degree four $\mathcal{H}_1,\ldots,\mathcal{H}_5$ on the complex projective plane. They are respectively described in affine chart by the following 1-forms*

1. $\omega_1 = y^4\mathrm{d}x - x^4\mathrm{d}y$;
2. $\omega_2 = y^3(2x-y)\mathrm{d}x + x^3(x-2y)\mathrm{d}y$;
3. $\omega_3 = y^2(6x^2+4xy+y^2)\mathrm{d}x - x^3(x+4y)\mathrm{d}y$;
4. $\omega_4 = y^3(4x+y)\mathrm{d}x + x^3(x+4y)\mathrm{d}y$;
5. $\omega_5 = y^2(6x^2+4xy+y^2)\mathrm{d}x + 3x^4\mathrm{d}y$.

By [11] we know that every foliation of degree $d\geq 1$ on $\mathbb{P}^2_{\mathbb{C}}$ can not have more than $3d$ (distinct) invariant lines. If this bound is reached for $\mathcal{F}\in\mathbf{F}(d)$, then $\mathcal{F}$ necessarily belongs to $\mathbf{FC}(d)$; in this case we say that $\mathcal{F}$ is *reduced convex*. To our knowledge the only reduced convex foliations known in the literature are those presented in [10, Table 1.1]: the FERMAT foliation $\mathcal{F}_0^d$ in any degree, the HESSE pencil $\mathcal{F}_H^4$ and the foliations given by the 1-forms

$$(y^2-1)(y^2-(\sqrt{5}-2)^2)(y+\sqrt{5}x)\mathrm{d}x - (x^2-1)(x^2-(\sqrt{5}-2)^2)(x+\sqrt{5}y)\mathrm{d}y,$$

$$(y^3-1)(y^3+7x^3+1)y\mathrm{d}x - (x^3-1)(x^3+7y^3+1)x\mathrm{d}y,$$

which have degrees 5 and 7 respectively. D. MARÍN and J. PEREIRA [10, Problem 9.1] asked the following question: are there other reduced convex foliations? The answer in degree 2, resp. 3, to this question is negative, by [9, Proposition 7.4], resp. [4, Corollary 6.9]. Theorem A allows us to show that the answer to [10, Problem 9.1] in degree 4 is also negative.

Theorem B. — *Up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$, the* FERMAT *foliation $\mathcal{F}_0^4$ and the* HESSE *pencil $\mathcal{F}_H^4$ are the only reduced convex foliations of degree four on $\mathbb{P}^2_{\mathbb{C}}$.*

1. Preliminaries

1.1. Singularities and inflection divisor of a foliation on the projective plane. — A degree d holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ is defined in homogeneous coordinates $[x:y:z]$ by a 1-form

$$\omega = a(x,y,z)\mathrm{d}x + b(x,y,z)\mathrm{d}y + c(x,y,z)\mathrm{d}z,$$

where a, b and c are homogeneous polynomials of degree $d+1$ without common factor and satisfying the EULER condition $i_{\mathrm{R}}\omega = 0$, where $\mathrm{R} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ denotes the radial vector field and i_{R} is the interior product by R. The *singular locus* $\mathrm{Sing}\mathcal{F}$ of $\mathcal{F}$ is the projectivization of the singular locus of ω

$$\mathrm{Sing}\,\omega = \{(x,y,z) \in \mathbb{C}^3 \,|\, a(x,y,z) = b(x,y,z) = c(x,y,z) = 0\}.$$

Let us recall some local notions attached to the pair $(\mathcal{F}, s)$, where $s \in \mathrm{Sing}\mathcal{F}$. The germ of $\mathcal{F}$ at s is defined, up to multiplication by a unity in the local ring $\mathcal{O}_s$ at s, by a vector field $\mathrm{X} = A(\mathrm{u},\mathrm{v})\frac{\partial}{\partial \mathrm{u}} + B(\mathrm{u},\mathrm{v})\frac{\partial}{\partial \mathrm{v}}$. The *vanishing order* $\nu(\mathcal{F}, s)$ of $\mathcal{F}$ at s is given by

$$\nu(\mathcal{F}, s) = \min\{\nu(A, s), \nu(B, s)\},$$

where $\nu(g, s)$ denotes the vanishing order of the function g at s. The *tangency order* of $\mathcal{F}$ with a generic line passing through s is the integer

$$\tau(\mathcal{F}, s) = \min\{k \geq \nu(\mathcal{F}, s) \,:\, \det(J_s^k\mathrm{X}, \mathrm{R}_s) \neq 0\},$$

where $J_s^k\mathrm{X}$ denotes the k-jet of X at s and R_s is the radial vector field centered at s. The MILNOR *number* of $\mathcal{F}$ at s is the integer

$$\mu(\mathcal{F}, s) = \dim_{\mathbb{C}} \mathcal{O}_s / \langle A, B\rangle,$$

where $\langle A, B\rangle$ denotes the ideal of $\mathcal{O}_s$ generated by A and B.

The singularity s is called *radial of order* $n-1$ if $\nu(\mathcal{F}, s) = 1$ and $\tau(\mathcal{F}, s) = n$.

The singularity s is called *non-degenerate* if $\mu(\mathcal{F}, s) = 1$, or equivalently if the linear part $J_s^1\mathrm{X}$ of X possesses two non-zero eigenvalues λ, μ. In this case, the quantity $\mathrm{BB}(\mathcal{F}, s) = \frac{\lambda}{\mu} + \frac{\mu}{\lambda} + 2$ is called the BAUM-BOTT *invariant* of $\mathcal{F}$ at s (*see* [2]). By [6] there is at least a germ of curve $\mathcal{C}$ at s which is invariant by $\mathcal{F}$. Up to local diffeomorphism we can assume that $s = (0,0)$ $\mathrm{T}_s\mathcal{C} = \{\mathrm{u} = 0\}$ and $J_s^1\mathrm{X} = \lambda\mathrm{u}\frac{\partial}{\partial \mathrm{u}} + (\varepsilon\mathrm{u} + \mu\mathrm{v})\frac{\partial}{\partial \mathrm{v}}$, where we can take $\varepsilon = 0$ if $\lambda \neq \mu$. The quantity $\mathrm{CS}(\mathcal{F}, \mathcal{C}, s) = \frac{\lambda}{\mu}$ is called the CAMACHO-SAD *index* of $\mathcal{F}$ at s along $\mathcal{C}$.

Let us also recall the notion of inflection divisor of $\mathcal{F}$. Let $\mathrm{Z} = E\frac{\partial}{\partial x} + F\frac{\partial}{\partial y} + G\frac{\partial}{\partial z}$ be a homogeneous vector field of degree d on $\mathbb{C}^3$ non collinear to the radial vector field describing $\mathcal{F}$, *i.e.* such that $\omega = i_{\mathrm{R}}i_{\mathrm{Z}}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. The *inflection divisor* of $\mathcal{F}$, denoted by $\mathrm{I}_{\mathcal{F}}$, is the divisor of $\mathbb{P}^2_{\mathbb{C}}$ defined by the homogeneous equation

$$\begin{vmatrix} x & E & \mathrm{Z}(E) \\ y & F & \mathrm{Z}(F) \\ z & G & \mathrm{Z}(G) \end{vmatrix} = 0.$$

This divisor has been studied in [11] in a more general context. In particular, the following properties has been proved.

1. On $\mathbb{P}^2_{\mathbb{C}} \setminus \operatorname{Sing}\mathcal{F}$, $\mathrm{I}_{\mathcal{F}}$ coincides with the curve described by the inflection points of the leaves of $\mathcal{F}$;
2. If $\mathcal{C}$ is an irreducible algebraic curve invariant by $\mathcal{F}$ then $\mathcal{C} \subset \mathrm{I}_{\mathcal{F}}$ if and only if $\mathcal{C}$ is an invariant line;
3. $\mathrm{I}_{\mathcal{F}}$ can be decomposed into $\mathrm{I}_{\mathcal{F}} = \mathrm{I}^{\mathrm{inv}}_{\mathcal{F}} + \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}}$, where the support of $\mathrm{I}^{\mathrm{inv}}_{\mathcal{F}}$ consists in the set of invariant lines of $\mathcal{F}$ and the support of $\mathrm{I}^{\mathrm{tr}}_{\mathcal{F}}$ is the closure of the isolated inflection points along the leaves of $\mathcal{F}$;
4. The degree of the divisor $\mathrm{I}_{\mathcal{F}}$ is $3d$.

The foliation $\mathcal{F}$ will be called *convex* if its inflection divisor $\mathrm{I}_{\mathcal{F}}$ is totally invariant by $\mathcal{F}$, *i.e.* if $\mathrm{I}_{\mathcal{F}}$ is a product of invariant lines.

1.2. Geometry of homogeneous foliations. — A foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ is said to be *homogeneous* if there is an affine chart (x,y) of $\mathbb{P}^2_{\mathbb{C}}$ in which it is invariant under the action of the group of homotheties $(x,y) \longmapsto \lambda(x,y)$, $\lambda \in \mathbb{C}^*$. Such a foliation $\mathcal{H}$ is then defined by a 1-form

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y,$$

where A and B are homogeneous polynomials of degree d without common factor. This 1-form writes in homogeneous coordinates as

$$zA(x,y)\mathrm{d}x + zB(x,y)\mathrm{d}y - (xA(x,y) + yB(x,y))\,\mathrm{d}z.$$

Thus the foliation $\mathcal{H}$ has at most $d+2$ singularities whose origin O of the affine chart $z=1$ is the only singular point of $\mathcal{H}$ which is not situated on the line at infinity $L_\infty = \{z=0\}$; moreover $\nu(\mathcal{H},O) = d$.
In the sequel we will assume that d is greater than or equal to 2. In this case the point O is the only singularity of $\mathcal{H}$ having vanishing order d.
We know from [4] that the inflection divisor of $\mathcal{H}$ is given by $z\mathrm{C}_{\mathcal{H}}\mathrm{D}_{\mathcal{H}} = 0$, where $\mathrm{C}_{\mathcal{H}} = xA + yB \in \mathbb{C}[x,y]_{d+1}$ denotes the *tangent cone* of $\mathcal{H}$ at the origin O and $\mathrm{D}_{\mathcal{H}} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x} \in \mathbb{C}[x,y]_{2d-2}$. From this we deduce that:

(i) the support of the divisor $\mathrm{I}^{\mathrm{inv}}_{\mathcal{H}}$ consists of the lines of the tangent cone $\mathrm{C}_{\mathcal{H}} = 0$ and the line at infinity L_∞;

(ii) the divisor $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}$ decomposes as $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}} = \prod_{i=1}^{n} T_i^{\rho_i - 1}$ for some number $n \leq \deg \mathrm{D}_{\mathcal{H}} = 2d-2$ of lines T_i passing through O, $\rho_i - 1$ being the inflection order of the line T_i.

Proposition 1.1 ([4], Proposition 2.2). — *With the previous notations, for any point $s \in \operatorname{Sing}\mathcal{H} \cap L_\infty$, we have*
1. $\nu(\mathcal{H},s) = 1$;
2. *the line joining the origin O to the point s is invariant by $\mathcal{H}$ and it appears with multiplicity $\tau(\mathcal{H},s) - 1$ in the divisor $\mathrm{D}_{\mathcal{H}} = 0$, i.e.*

$$\mathrm{D}_{\mathcal{H}} = \mathrm{I}^{\mathrm{tr}}_{\mathcal{H}} \prod_{s \in \operatorname{Sing}\mathcal{H} \cap L_\infty} L_s^{\tau(\mathcal{H},s)-1}.$$

Definition 1.2 ([4]). — Let $\mathcal{H}$ be a homogeneous foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ having a certain number $m \leq d+1$ of radial singularities s_i of order $\tau_i - 1$, $2 \leq \tau_i \leq d$ for $i = 1,2,\ldots,m$. The support of the divisor $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}$ consists of a certain number $n \leq 2d-2$ of transverse inflection lines T_j of order $\rho_j - 1$, $2 \leq \rho_j \leq d$ for $j = 1,2,\ldots,n$. We define the *type of the foliation* $\mathcal{H}$ by

$$\mathcal{T}_{\mathcal{H}} = \sum_{i=1}^{m} \mathrm{R}_{\tau_i - 1} + \sum_{j=1}^{n} \mathrm{T}_{\rho_j - 1} = \sum_{k=1}^{d-1} (r_k \cdot \mathrm{R}_k + t_k \cdot \mathrm{T}_k) \in \mathbb{Z}\left[\mathrm{R}_1, \mathrm{R}_2, \ldots, \mathrm{R}_{d-1}, \mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_{d-1}\right].$$

Example 1.3. — Let us consider the homogeneous foliation $\mathcal{H}$ of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$ defined by

$$\omega = y^5\mathrm{d}x + 2x^3(3x^2 - 5y^2)\mathrm{d}y.$$

A straightforward computation leads to

$$\mathrm{C}_{\mathcal{H}} = xy\left(6x^4 - 10x^2y^2 + y^4\right) \qquad \text{and} \qquad \mathrm{D}_{\mathcal{H}} = 150x^2y^4(x-y)(x+y).$$

We see that the set of radial singularities of $\mathcal{H}$ consists of the two points $s_1 = [0:1:0]$ and $s_2 = [1:0:0]$; their orders of radiality are equal to 2 and 4 respectively. Moreover the support of the divisor $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}$ is the union of the two lines $x-y=0$ and $x+y=0$; they are transverse inflection lines of order 1. Therefore the foliation $\mathcal{H}$ is of type $\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_2 + 1\cdot\mathrm{R}_4 + 2\cdot\mathrm{T}_1$.

Following [4] to every homogeneous foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ is associated a rational map from the RIEMANN sphere $\mathbb{P}^1_{\mathbb{C}}$ to itself of degree d denoted by $\underline{\mathcal{G}}_{\mathcal{H}}$ and defined as follows: if $\mathcal{H}$ is described by $\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y$, with A and B being homogeneous polynomials of degree d without common factor, the image of the point $[x:y] \in \mathbb{P}^1_{\mathbb{C}}$ by $\underline{\mathcal{G}}_{\mathcal{H}}$ is the point $[-A(x,y):B(x,y)] \in \mathbb{P}^1_{\mathbb{C}}$. It is clear that this definition does not depend on the choice of the homogeneous 1-form ω describing the foliation $\mathcal{H}$. Notice that the map $\underline{\mathcal{G}}_{\mathcal{H}}$ has the following properties (*see* [4]):

1. the fixed points of $\underline{\mathcal{G}}_{\mathcal{H}}$ correspond to the tangent cone of $\mathcal{H}$ at the origin O (*i.e.* $[a:b] \in \mathbb{P}^1_{\mathbb{C}}$ is fixed by $\underline{\mathcal{G}}_{\mathcal{H}}$ if and only if the line $by - ax = 0$ is invariant by $\mathcal{H}$);
2. the point $[a:b] \in \mathbb{P}^1_{\mathbb{C}}$ is a fixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$ if and only if the point $[b:a:0] \in L_\infty$ is a radial singularity of $\mathcal{H}$. The multiplicity of the critical point $[a:b]$ of $\underline{\mathcal{G}}_{\mathcal{H}}$ is exactly equal to the the radiality order of the singularity at infinity;
3. the point $[a:b] \in \mathbb{P}^1_{\mathbb{C}}$ is a non-fixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$ if and only if the line $by - ax = 0$ is a transverse inflection line of $\mathcal{H}$. The multiplicity of the critical point $[a:b]$ of $\underline{\mathcal{G}}_{\mathcal{H}}$ is precisely equal to the inflection order of this line.

It follows, in particular, that a homogeneous foliation $\mathcal{H}$ on $\mathbb{P}^2_{\mathbb{C}}$ is convex if and only if its associated map $\underline{\mathcal{G}}_{\mathcal{H}}$ has only fixed critical points; more precisely, a homogeneous foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ is convex of type $\mathcal{T}_{\mathcal{H}} = \sum_{k=1}^{d-1} r_k\cdot\mathrm{R}_k$ if and only if the map $\underline{\mathcal{G}}_{\mathcal{H}}$ possesses r_1, resp. $r_2, \ldots$, resp. r_{d-1} fixed critical points of multiplicity 1, resp. $2\ldots$, resp. $d-1$, with $\sum_{k=1}^{d-1} kr_k = 2d-2$.
For recent results on rational self-maps of $\mathbb{P}^1_{\mathbb{C}}$ with only fixed critical points, we refer to [8].

2. Proof of Theorem A

Before proving Theorem A, let us recall the next result which follows from Propositions 4.1 and 4.2 of [4]:

Proposition 2.1 ([4]). — *Let $\mathcal{H}$ be a convex homogeneous foliation of degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$. Let ν be an integer between 1 and $d-2$. Then, $\mathcal{H}$ is of type*

$$\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_{d-1}, \qquad \textit{resp. } \mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_\nu + 1\cdot\mathrm{R}_{d-\nu-1} + 1\cdot\mathrm{R}_{d-1},$$

if and only if it is linearly conjugated to the foliation $\mathcal{H}_1^d$, resp. $\mathcal{H}_3^{d,\nu}$ given by

$$\omega_1^d = y^d\mathrm{d}x - x^d\mathrm{d}y, \qquad \textit{resp. } \omega_3^{d,\nu} = \sum_{i=\nu+1}^{d}\binom{d}{i}x^{d-i}y^i\mathrm{d}x - \sum_{i=0}^{\nu}\binom{d}{i}x^{d-i}y^i\mathrm{d}y.$$

Proof of Theorem A. — Let $\mathcal{H}$ be a convex homogeneous foliation of degree 4 on $\mathbb{P}^2_{\mathbb{C}}$, defined in the affine chart (x,y), by the 1-form

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y, \quad A,B \in \mathbb{C}[x,y]_4, \;\; \gcd(A,B)=1.$$

CAMACHO-SAD index theorem implies, *cf.* [5, Remark 2.5], that the foliation $\mathcal{H}$ can not have $4+1=5$ distinct radial singularities, in other words it can not be of type $4\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2$. We are then in one of the following situations:

$$\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_3\,; \qquad \mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3\,; \qquad \mathcal{T}_{\mathcal{H}} = 3\cdot\mathrm{R}_2\,;$$
$$\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2\,; \qquad \mathcal{T}_{\mathcal{H}} = 3\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3.$$

- If $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_3$, resp. $\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$, then by [4, Propositions 4.1, 4.2], the 1-form ω is linearly conjugated to
$$\omega_1^4 = y^4\mathrm{d}x - x^4\mathrm{d}y = \omega_1,$$
$$\text{resp. } \omega_3^{4,1} = \sum_{i=2}^{4}\binom{4}{i}x^{4-i}y^i\mathrm{d}x - \sum_{i=0}^{1}\binom{4}{i}x^{4-i}y^i\mathrm{d}y = y^2(6x^2+4xy+y^2)\mathrm{d}x - x^3(x+4y)\mathrm{d}y = \omega_3.$$
- Assume that $\mathcal{T}_{\mathcal{H}} = 3\cdot\mathrm{R}_2$. This means that the rational map $\underline{\mathcal{G}}_{\mathcal{H}} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, $\underline{\mathcal{G}}_{\mathcal{H}}(z) = -\dfrac{A(1,z)}{B(1,z)}$, admits three different fixed critical points of multiplicity 2. By [8, page 79], $\underline{\mathcal{G}}_{\mathcal{H}}$ is conjugated by a MÖBIUS transformation to $z \mapsto -\dfrac{z^3(2-z)}{1-2z}$. As a consequence, ω is linearly conjugated to
$$\omega_2 = y^3(2x-y)\mathrm{d}x + x^3(x-2y)\mathrm{d}y.$$
- Assume that $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2$. Then the rational map $\underline{\mathcal{G}}_{\mathcal{H}}$ possesses four fixed critical points, two of them having multiplicity 1 and the other two having multiplicity 2. This implies, by [8, page 79], that up to conjugation by a MÖBIUS transformation, $\underline{\mathcal{G}}_{\mathcal{H}}$ writes as
$$z \mapsto -\frac{z^3(2z+3cz-4c-3)}{z+c},$$
where $c = -3/8 \pm \sqrt{5}/8$. Thus, up to linear conjugation
$$\omega = y^3(2y+3cy-4cx-3x)\mathrm{d}x + x^3(y+cx)\mathrm{d}y, \qquad c = -\frac{3}{8} \pm \frac{\sqrt{5}}{8}.$$
In both cases ($c = -3/8+\sqrt{5}/8$ or $c = -3/8-\sqrt{5}/8$), the 1-form ω is linearly conjugated to
$$\omega_4 = y^3(4x+y)\mathrm{d}x + x^3(x+4y)\mathrm{d}y\,.$$
Indeed,
$$\omega_4 = \frac{3c+2}{2}\varphi^*\omega, \quad \text{where } \varphi = (2x, 8cy).$$
- Finally, consider the last situation: $\mathcal{T}_{\mathcal{H}} = 3\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3$. Up to linear conjugation we can assume that $\mathrm{D}_{\mathcal{H}} = cx^3y(y-x)(y-\alpha x)$ and $\mathrm{C}_{\mathcal{H}}(0,1) = \mathrm{C}_{\mathcal{H}}(1,0) = \mathrm{C}_{\mathcal{H}}(1,1) = \mathrm{C}_{\mathcal{H}}(1,\alpha) = 0$, for some $c,\alpha \in \mathbb{C}^*, \alpha \neq 1$. The points $\infty = [1:0]$, $[0:1]$, $[1:1]$, $[1:\alpha] \in \mathbb{P}^1_{\mathbb{C}}$ are then fixed and critical for $\underline{\mathcal{G}}_{\mathcal{H}}$, having respective multiplicities $3,1,1,1$. By [4, Lemma 3.9], there exist constants $a_0,a_2,b \in \mathbb{C}^*, a_1 \in \mathbb{C}$ such that
$$B(x,y) = bx^4, \quad A(x,y) = (a_0x^2+a_1xy+a_2y^2)y^2, \quad (z-1)^2 \text{ divides } P(z), \quad (z-\alpha)^2 \text{ divides } Q(z),$$

where $P(z) := A(1,z)+B(1,z)$ and $Q(z) := A(1,z)+\alpha B(1,z)$. It follows that

$$\begin{cases} P(1)=0 \\ P'(1)=0 \\ Q(\alpha)=0 \\ Q'(\alpha)=0 \end{cases} \Leftrightarrow \begin{cases} a_0+a_1+a_2+b=0 \\ 2a_0+3a_1+4a_2=0 \\ a_2\alpha^3+a_1\alpha^2+a_0\alpha+b=0 \\ 4a_2\alpha^2+3a_1\alpha+2a_0=0 \end{cases} \Leftrightarrow \begin{cases} a_0=2a_2\alpha \\ a_1=-\dfrac{4a_2(\alpha+1)}{3} \\ b=-\dfrac{a_2(2\alpha-1)}{3} \\ \alpha^2-\alpha+1=0 \end{cases}$$

By replacing ω by $\dfrac{3}{a_2}\omega$, we can assume that

$$\omega = y^2(6\alpha x^2-4(\alpha+1)xy+3y^2)\mathrm{d}x-(2\alpha-1)x^4\mathrm{d}y, \qquad \alpha^2-\alpha+1=0.$$

The 1-form ω is linearly conjugated to

$$\omega_5 = y^2(6x^2+4xy+y^2)\mathrm{d}x+3x^4\mathrm{d}y\,.$$

Indeed, the fact that α satisfies $\alpha^2-\alpha+1=0$ implies that

$$\omega_5 = \frac{1-\alpha}{(\alpha-2)^3}\varphi^*\omega, \quad \text{where } \varphi = \Big((\alpha-2)x, y\Big).$$

The foliations $\mathcal{H}_i, i=1,\ldots,5$, are not linearly conjugated because, by construction, $\mathcal{T}_{\mathcal{H}_j} \neq \mathcal{T}_{\mathcal{H}_i}$ for each $j\neq i$. This ends the proof of the theorem. □

A remarkable feature of the classification obtained is that all the singularities of the foliations $\mathcal{H}_i, i=1,\ldots,5$, on the line at infinity are non-degenerated. In the following section we will need the values of the CAMACHO-SAD indices $\mathrm{CS}(\mathcal{H}_i, L_\infty, s)$, $s\in \mathrm{Sing}\mathcal{H}_i\cap L_\infty$. For this reason, we have computed, for each $i=1,\ldots,5$, the following polynomial (called CAMACHO-SAD *polynomial of the homogeneous foliation* $\mathcal{H}_i$)

$$\mathrm{CS}_{\mathcal{H}_i}(\lambda) = \prod_{s\in\mathrm{Sing}\mathcal{H}_i\cap L_\infty} (\lambda - \mathrm{CS}(\mathcal{H}_i, L_\infty, s)).$$

The following table summarizes the types and the CAMACHO-SAD polynomials of the foliations $\mathcal{H}_i$, $i = 1,\ldots,5$.

i	$\mathcal{T}_{\mathcal{H}_i}$	$\mathrm{CS}_{\mathcal{H}_i}(\lambda)$
1	$2\cdot\mathrm{R}_3$	$(\lambda-1)^2(\lambda+\frac{1}{3})^3$
2	$3\cdot\mathrm{R}_2$	$(\lambda-1)^3(\lambda+1)^2$
3	$1\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$	$(\lambda-1)^3(\lambda+\frac{13+2\sqrt{13}}{13})(\lambda+\frac{13-2\sqrt{13}}{13})$
4	$2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2$	$(\lambda-1)^4(\lambda+3)$
5	$3\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3$	$(\lambda-1)^4(\lambda+3)$

TABLE 1. Types and CAMACHO-SAD polynomials of the homogeneous foliations given by Theorem A.

3. Proof of Theorem B

The proof of Theorem B is based on the classification of convex homogeneous foliations of degree four on $\mathbb{P}^2_{\mathbb{C}}$ given by Theorem A and on the three following results which hold in arbitrary degree.
First, notice that if $\mathcal{F}$ is a foliation of degree $d \geq 1$ on $\mathbb{P}^2_{\mathbb{C}}$ and if s is a singular point of $\mathcal{F}$ then

$$\sigma(\mathcal{F}, s) \leq \tau(\mathcal{F}, s) + 1 \leq d + 1,$$

where $\sigma(\mathcal{F}, s)$ denotes the number of (distinct) invariant lines of $\mathcal{F}$ passing through s.
The following lemma shows that the left-hand inequality above is an equality in the case where $\mathcal{F}$ is a reduced convex foliation.

Lemma 3.1. — *Let $\mathcal{F}$ be a reduced convex foliation of degree $d \geq 1$ on $\mathbb{P}^2_{\mathbb{C}}$. Then, through each singular point s of $\mathcal{F}$ pass exactly $\tau(\mathcal{F}, s) + 1$ invariant lines of $\mathcal{F}$, i.e. $\sigma(\mathcal{F}, s) = \tau(\mathcal{F}, s) + 1$.*

Proof. — Let s be a singular point of $\mathcal{F}$. Since the inflection divisor $\mathrm{I}_{\mathcal{F}}$ of $\mathcal{F}$ is totally invariant by $\mathcal{F}$ and it is reduced, we deduce that $\mu(\mathcal{F}, s) = 1$ ([4, Lemma 6.8]) and the number $\sigma(\mathcal{F}, s)$ coincides with the vanishing order of $\mathrm{I}_{\mathcal{F}}$ at s. On the other hand, an elementary computation, using the equality $\mu(\mathcal{F}, s) = 1$, shows that the vanishing order of $\mathrm{I}_{\mathcal{F}}$ at s is equal to $\tau(\mathcal{F}, s) + 1$. Hence the lemma holds. □

The following result allows us to reduce the study of the convexity to the homogeneous framework:

Proposition 3.2. — *Let $\mathcal{F}$ be a reduced convex foliation of degree $d \geq 1$ on $\mathbb{P}^2_{\mathbb{C}}$ and let ℓ be one of its $3d$ invariant lines. There is a convex homogeneous foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ satisfying the following properties:*

- *(i)* $\mathcal{H} \in \overline{O(\mathcal{F})}$;
- *(ii)* ℓ *is invariant by* $\mathcal{H}$;
- *(iii)* $\mathrm{Sing}\mathcal{H} \cap \ell = \mathrm{Sing}\mathcal{F} \cap \ell$;
- *(iv)* $\forall\, s \in \mathrm{Sing}\mathcal{H} \cap \ell,\ \mu(\mathcal{H}, s) = 1$;
- *(v)* $\forall\, s \in \mathrm{Sing}\mathcal{H} \cap \ell,\ \tau(\mathcal{H}, s) = \tau(\mathcal{F}, s)$;
- *(vi)* $\forall\, s \in \mathrm{Sing}\mathcal{H} \cap \ell,\ \mathrm{CS}(\mathcal{H}, \ell, s) = \mathrm{CS}(\mathcal{F}, \ell, s)$.

Proof. — We take a homogeneous coordinate system $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ such that $\ell = \{z = 0\}$. Since ℓ is $\mathcal{F}$-invariant, $\mathcal{F}$ is defined in the affine chart $z = 1$ by a 1-form of the following type

$$\omega = \sum_{i=0}^{d} (A_i(x, y)\mathrm{d}x + B_i(x, y)\mathrm{d}y),$$

where A_i, B_i are homogeneous polynomials of degree i. Using the fact that every reduced convex foliation on $\mathbb{P}^2_{\mathbb{C}}$ has only non-degenerate singularities ([4, Lemma 6.8]) and arguing as in the proof of [4, Proposition 6.4], we see that the 1-form $\omega_d = A_d(x, y)\mathrm{d}x + B_d(x, y)\mathrm{d}y$ defines a homogeneous foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$, and that this foliation satisfies the announced properties (i), (ii), (iii), (iv) and (vi). In particular, since $\mathcal{F}$ is convex by hypothesis, property (i) implies that $\mathcal{H}$ is also convex.
Let us show that $\mathcal{H}$ also satisfies property (v). Set $\Lambda := \mathrm{Sing}\mathcal{H} \cap \ell = \mathrm{Sing}\mathcal{F} \cap \ell$; since $\mathcal{F}$ possesses $3d$ invariant lines, we have $\sum_{s \in \Lambda} \big(\sigma(\mathcal{F}, s) - 1\big) = 3d - 1$. By Lemma 3.1, this is equivalent to $\sum_{s \in \Lambda} \tau(\mathcal{F}, s) = 3d - 1$. By [4, Proposition 2.2] the convexity of $\mathcal{H}$ implies that $\sum_{s \in \Lambda} \tau(\mathcal{H}, s) = 2d - 2$. Moreover, the already proved property (iv) ensures that $\#\Lambda = d + 1$. It follows that

$$\sum_{s \in \Lambda} \big(\tau(\mathcal{F}, s) - 1\big) = (3d - 1) - \#\Lambda = 2d - 2 = \sum_{s \in \Lambda} \big(\tau(\mathcal{H}, s) - 1\big).$$

Thus, in order to see that $\mathcal{H}$ satisfies property (v), it is enough to prove that $\tau(\mathcal{F},s)\leq\tau(\mathcal{H},s)$ for each $s\in\Lambda$. Let us fix $s\in\Lambda$. Up to conjugating ω by a linear isomorphism of $\mathbb{C}^2=(z=1)$, we can assume that $s=[0:1:0]$. The foliations $\mathcal{F}$ and $\mathcal{H}$ are respectively defined in the affine chart $y=1$ by the 1-forms

$$\theta=\sum_{i=0}^{d}z^{d-i}[A_i(x,1)(z\mathrm{d}x-x\mathrm{d}z)-B_i(x,1)\mathrm{d}z]\qquad\text{and}\qquad\theta_d=A_d(x,1)(z\mathrm{d}x-x\mathrm{d}z)-B_d(x,1)\mathrm{d}z.$$

As a consequence

$$\tau(\mathcal{F},s)=\min\Big\{k\geq1\,:\,J^k_{(0,0)}\Big(\sum_{i=0}^{d}z^{d-i}B_i(x,1)\Big)\neq0\Big\}\leq\min\Big\{k\geq1\,:\,J^k_0\big(B_d(x,1)\big)\neq0\Big\}=\tau(\mathcal{H},s).$$

□

Remark 3.3. — If $\mathcal{F}$ is a foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ then (*see* [3])

$$\sum_{s\in\mathrm{Sing}\mathcal{F}}\mu(\mathcal{F},s)=d^2+d+1\qquad\text{and}\qquad\sum_{s\in\mathrm{Sing}\mathcal{F}}\mathrm{BB}(\mathcal{F},s)=(d+2)^2.\tag{3.1}$$

Lemma 3.4. — *Every foliation of degree $d\geq1$ on $\mathbb{P}^2_{\mathbb{C}}$ possesses at least a non radial singularity.*

This lemma follows from the formulas (3.1) and the obvious following remark: if a foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ admits a radial singularity s, then $\mu(\mathcal{F},s)=1$ and $\mathrm{BB}(\mathcal{F},s)=4$.

Proof of Theorem B. — Let $\mathcal{F}$ be a reduced convex foliation of degree 4 on $\mathbb{P}^2_{\mathbb{C}}$. Let us denote by Σ the set of non radial singularities of $\mathcal{F}$. By Lemma 3.4, Σ is nonempty. Since by hypothesis $\mathcal{F}$ is reduced convex, all its singularities have MILNOR number 1 ([4, Lemma 6.8]). The set Σ consists then of the singularities $s\in\mathrm{Sing}\mathcal{F}$ such that $\tau(\mathcal{F},s)=1$. Let m be a point of Σ; by Lemma 3.1, through the point m pass exactly two $\mathcal{F}$-invariant lines $\ell_m^{(1)}$ and $\ell_m^{(2)}$.
On the other hand, for any line ℓ invariant by $\mathcal{F}$, Proposition 3.2 ensures the existence of a convex homogeneous foliation $\mathcal{H}_\ell$ of degree 4 on $\mathbb{P}^2_{\mathbb{C}}$ belonging to $\overline{O(\mathcal{F})}$ and such that the line ℓ is $\mathcal{H}_\ell$-invariant. Therefore $\mathcal{H}_\ell$, and in particular each $\mathcal{H}_{\ell_m^{(i)}}$, is linearly conjugated to one of the five homogeneous foliations given by Theorem A. Proposition 3.2 also ensures that

(𝔞) $\mathrm{Sing}\mathcal{F}\cap\ell=\mathrm{Sing}\mathcal{H}_\ell\cap\ell$;
(𝔟) $\forall\,s\in\mathrm{Sing}\mathcal{H}_\ell\cap\ell,\ \mu(\mathcal{H}_\ell,s)=1$;
(𝔠) $\forall\,s\in\mathrm{Sing}\mathcal{H}_\ell\cap\ell,\ \tau(\mathcal{H}_\ell,s)=\tau(\mathcal{F},s)$;
(𝔡) $\forall\,s\in\mathrm{Sing}\mathcal{H}_\ell\cap\ell,\ \mathrm{CS}(\mathcal{H}_\ell,\ell,s)=\mathrm{CS}(\mathcal{F},\ell,s)$.

Since $\mathrm{CS}(\mathcal{F},\ell_m^{(1)},m)\mathrm{CS}(\mathcal{F},\ell_m^{(2)},m)=1$, relation (𝔡) implies that $\mathrm{CS}(\mathcal{H}_{\ell_m^{(1)}},\ell_m^{(1)},m)\mathrm{CS}(\mathcal{H}_{\ell_m^{(2)}},\ell_m^{(2)},m)=1$. This equality and Table 1 lead to

$$\{\mathrm{CS}(\mathcal{H}_{\ell_m^{(1)}},\ell_m^{(1)},m),\,\mathrm{CS}(\mathcal{H}_{\ell_m^{(2)}},\ell_m^{(2)},m)\}=\{-3,-\tfrac{1}{3}\}\quad\text{or}\quad\mathrm{CS}(\mathcal{H}_{\ell_m^{(1)}},\ell_m^{(1)},m)=\mathrm{CS}(\mathcal{H}_{\ell_m^{(2)}},\ell_m^{(2)},m)=-1.$$

At first let us suppose that it is possible to choose $m\in\Sigma$ so that

$$\{\mathrm{CS}(\mathcal{H}_{\ell_m^{(1)}},\ell_m^{(1)},m),\,\mathrm{CS}(\mathcal{H}_{\ell_m^{(2)}},\ell_m^{(2)},m)\}=\{-3,-\tfrac{1}{3}\}.$$

By renumbering the $\ell_m^{(i)}$ we can assume that $\mathrm{CS}(\mathcal{H}_{\ell_m^{(1)}},\ell_m^{(1)},m)=-\frac{1}{3}$ and $\mathrm{CS}(\mathcal{H}_{\ell_m^{(2)}},\ell_m^{(2)},m)=-3$. Consulting Table 1, we see that

$$\mathcal{T}_{\mathcal{H}_{\ell_m^{(1)}}}=2\cdot\mathrm{R}_3,\qquad\qquad\mathcal{T}_{\mathcal{H}_{\ell_m^{(2)}}}\in\Big\{3\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3,\,2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2\Big\}.$$

Therefore, it follows from relations $(\mathfrak{a})$ and $(\mathfrak{c})$ that $\mathcal{F}$ possesses two radial singularities m_1, m_2 of order 3 on the line $\ell_m^{(1)}$ and a radial singularity m_3 of order 2 or 3 on the line $\ell_m^{(2)}$.
We will see that the radiality order of the singularity m_3 of $\mathcal{F}$ is necessarily 3, *i.e.* $\tau(\mathcal{F}, m_3) = 4$. By [3, Proposition 2, page 23], the fact that $\tau(\mathcal{F}, m_1) + \tau(\mathcal{F}, m_3) \geq 4 + 3 > \deg \mathcal{F}$ implies the invariance by $\mathcal{F}$ of the line $\ell = (m_1 m_3)$; if $\tau(\mathcal{F}, m_3)$ were equal to 3, then relations $(\mathfrak{a})$, $(\mathfrak{b})$ and $(\mathfrak{c})$, combined with the convexity of the foliation $\mathcal{H}_\ell$, would imply that $\mathcal{T}_{\mathcal{H}_\ell} = 1 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{R}_2 + 1 \cdot \mathrm{R}_3$ so that (*see* Table 1) $\mathcal{H}_\ell$ would possess a singularity m' on the line ℓ satisfying $\mathrm{CS}(\mathcal{H}_\ell, \ell, m') \in \{-\frac{13+2\sqrt{13}}{13}, -\frac{13-2\sqrt{13}}{13}\}$ which is not possible.
By construction, the three points m_1, m_2 and m_3 are not aligned. We have thus shown that $\mathcal{F}$ admits three non-aligned radial singularities of order 3. By [4, Proposition 6.3] the foliation $\mathcal{F}$ is linearly conjugated to the FERMAT foliation $\mathcal{F}_0^4$.

Let us now consider the eventuality $\mathrm{CS}(\mathcal{H}_{\ell_m^{(1)}}, \ell_m^{(1)}, m) = \mathrm{CS}(\mathcal{H}_{\ell_m^{(2)}}, \ell_m^{(2)}, m) = -1$ for any choice of $m \in \Sigma$. In this case, Table 1 leads to $\mathcal{T}_{\mathcal{H}_{\ell_m^{(i)}}} = 3 \cdot \mathrm{R}_2$ for $i = 1, 2$. Then, as before, by using relations $(\mathfrak{a})$, $(\mathfrak{b})$ and $(\mathfrak{c})$, we obtain that $\mathcal{F}$ possesses exactly three radial singularities of order 2 on each line $\ell_m^{(i)}$. Moreover, every line joining a radial singularity of order 2 of $\mathcal{F}$ on $\ell_m^{(1)}$ and a radial singularity of order 2 of $\mathcal{F}$ on $\ell_m^{(2)}$ must contain necessarily a third radial singularity of order 2 of $\mathcal{F}$. We can then choose a homogeneous coordinate system $[x:y:z] \in \mathbb{P}^2_{\mathbb{C}}$ so that the points $m_1 = [0:0:1]$, $m_2 = [1:0:0]$ and $m_3 = [0:1:0]$ are radial singularities of order 2 of $\mathcal{F}$. Moreover, in this coordinate system the lines $x = 0$, $y = 0$, $z = 0$ must be invariant by $\mathcal{F}$ and there exist $x_0, y_0, z_0 \in \mathbb{C}^*$ such that the points $m_4 = [x_0 : 0 : 1]$, $m_5 = [1 : y_0 : 0]$, $m_6 = [0 : 1 : z_0]$ are radial singularities of order 2 of $\mathcal{F}$. The equalities $\nu(\mathcal{F}, m_1) = 1$, $\tau(\mathcal{F}, m_1) = 3$ and the invariance of the line $z = 0$ by $\mathcal{F}$ ensure that every 1-form ω defining $\mathcal{F}$ in the affine chart $z = 1$ is of type

$$\begin{aligned}\omega = {} & (x\mathrm{d}y - y\mathrm{d}x)(\gamma + c_0 x + c_1 y) + (\alpha_0 x^3 + \alpha_1 x^2 y + \alpha_2 x y^2 + \alpha_3 y^3)\mathrm{d}x + (\beta_0 x^3 + \beta_1 x^2 y + \beta_2 x y^2 + \beta_3 y^3)\mathrm{d}y \\ & + (a_0 x^4 + a_1 x^3 y + a_2 x^2 y^2 + a_3 x y^3 + a_4 y^4)\mathrm{d}x + (b_0 x^4 + b_1 x^3 y + b_2 x^2 y^2 + b_3 x y^3 + b_4 y^4)\mathrm{d}y,\end{aligned}$$

where $a_i, b_i, c_j, \alpha_k, \beta_k \in \mathbb{C}$ and $\gamma \in \mathbb{C}^*$.
In the affine chart $x = 1$, resp. $y = 1$, the foliation $\mathcal{F}$ is given by

$$\begin{aligned}\theta = {} & z^3(\gamma z + c_0 + c_1 y)\mathrm{d}y - (\alpha_0 z + \alpha_1 y z + \alpha_2 y^2 z + \alpha_3 y^3 z + a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4)\mathrm{d}z \\ & - (\beta_0 z + \beta_1 y z + \beta_2 y^2 z + \beta_3 y^3 z + b_0 + b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4)(y\mathrm{d}z - z\mathrm{d}y), \\ \text{resp. } \eta = {} & -z^3(\gamma z + c_0 x + c_1)\mathrm{d}x - (\beta_0 x^3 z + \beta_1 x^2 z + \beta_2 x z + \beta_3 z + b_0 x^4 + b_1 x^3 + b_2 x^2 + b_3 x + b_4)\mathrm{d}z \\ & + (\alpha_0 x^3 z + \alpha_1 x^2 z + \alpha_2 x z + \alpha_3 z + a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4)(z\mathrm{d}x - x\mathrm{d}z).\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}&\Big(J^2_{(y,z)=(0,0)}\theta\Big) \wedge \Big(y\mathrm{d}z - z\mathrm{d}y\Big) = -zP(y,z)\mathrm{d}y \wedge \mathrm{d}z, && \Big(J^2_{(x,z)=(0,0)}\eta\Big) \wedge \Big(z\mathrm{d}x - x\mathrm{d}z\Big) = zQ(x,z)\mathrm{d}x \wedge \mathrm{d}z, \\ &\Big(J^2_{(x,y)=(x_0,0)}\omega\Big) \wedge \Big((x - x_0)\mathrm{d}y - y\mathrm{d}x\Big) = x_0 R(x,y)\mathrm{d}x \wedge \mathrm{d}y, && \Big(J^2_{(y,z)=(y_0,0)}\theta\Big) \wedge \Big((y - y_0)\mathrm{d}z - z\mathrm{d}y\Big) = -zS(y,z)\mathrm{d}y \wedge \mathrm{d}z, \\ &\Big(J^2_{(x,z)=(0,z_0)}\eta\Big) \wedge \Big((z - z_0)\mathrm{d}x - x\mathrm{d}z\Big) = T(x,z)\mathrm{d}x \wedge \mathrm{d}z\end{aligned}$$

with

$P(y,z) = a_0 + a_1 y + \alpha_0 z + a_2 y^2 + \alpha_1 yz,$

$Q(x,z) = b_4 + b_3 x + \beta_3 z + b_2 x^2 + \beta_2 xz,$

$$\begin{aligned}R(x,y) &= -x_0^3(\alpha_0 + 3a_0x_0) + x_0^2(4\alpha_0 + 11a_0x_0)x + (\gamma + \alpha_1 x_0^2 + \beta_0 x_0^2 + 2a_1x_0^3 + 3b_0x_0^3)y - 2x_0(3\alpha_0 + 7a_0x_0)x^2 \\ &\quad + (c_0 - 3\alpha_1 x_0 - 3\beta_0 x_0 - 5a_1x_0^2 - 8b_0x_0^2)xy + (c_1 - \alpha_2 x_0 - \beta_1 x_0 - a_2x_0^2 - 2b_1x_0^2)y^2 + (3\alpha_0 + 6a_0x_0)x^3 \\ &\quad + (2\alpha_1 + 3\beta_0 + 3a_1x_0 + 6b_0x_0)x^2y + (\alpha_2 + 2\beta_1 + a_2x_0 + 3b_1x_0)xy^2 + (\beta_2 + b_2x_0)y^3,\end{aligned}$$

$$\begin{aligned}S(y,z) &= a_0 + b_0y_0 + a_3y_0^3 + 3a_4y_0^4 + b_3y_0^4 + 3b_4y_0^5 + (a_1 + b_1y_0 - 3a_3y_0^2 - 8a_4y_0^3 - 3b_3y_0^3 - 8b_4y_0^4)y \\ &\quad + (\alpha_0 + \beta_0 y_0 - \alpha_2 y_0^2 - 2\alpha_3 y_0^3 - \beta_2 y_0^3 - 2\beta_3 y_0^4)z + (a_2 + 3a_3y_0 + b_2y_0 + 6a_4y_0^2 + 3b_3y_0^2 + 6b_4y_0^3)y^2 \\ &\quad + (\alpha_1 + 2\alpha_2 y_0 + \beta_1 y_0 + 3\alpha_3 y_0^2 + 2\beta_2 y_0^2 + 3\beta_3 y_0^3)yz,\end{aligned}$$

$$\begin{aligned}T(x,z) &= -b_4z_0 - z_0(a_4 + b_3 - c_1z_0^2 - 3\gamma z_0^3)x + (b_4 - \beta_3 z_0)z - z_0(a_3 + b_2 + \beta_1 z_0 + 2c_0z_0^2)x^2 + (\beta_2 + 3c_1z_0 + 6\gamma z_0^2)xz^2 \\ &\quad + (b_3 - \alpha_3 z_0 - \beta_2 z_0 - 3c_1z_0^2 - 8\gamma z_0^3)xz + \beta_3 z^2 - z_0(a_2 + \alpha_1 z_0)x^3 + (b_2 - \alpha_2 z_0 + \beta_1 z_0 + 3c_0z_0^2)x^2z,\end{aligned}$$

so that the equality $\tau(\mathcal{F},m_2) = 3$ (resp. $\tau(\mathcal{F},m_3) = 3$, resp. $\tau(\mathcal{F},m_4) = 3$, resp. $\tau(\mathcal{F},m_5) = 3$, resp. $\tau(\mathcal{F},m_6) = 3$) implies that the polynomial P (resp. Q, resp. R, resp. S, resp. T) is identically zero. From $P = Q = 0$ we obtain $a_0 = a_1 = a_2 = \alpha_0 = \alpha_1 = b_4 = b_3 = b_2 = \beta_3 = \beta_2 = 0$. Next, from the equalities $R = S = T = 0$ we deduce that

$$\begin{aligned}&c_0 = 2\gamma y_0z_0(x_0y_0z_0 + 1), \quad c_1 = -2\gamma z_0, \quad \alpha_2 = 2\gamma y_0z_0^2(x_0y_0z_0 + 2), \quad \alpha_3 = -2\gamma z_0^2, \quad \beta_0 = 2\gamma x_0y_0^3z_0^3, \\ &\beta_1 = -2\gamma y_0z_0^2(2x_0y_0z_0 + 1), \quad a_3 = -2\gamma y_0z_0^3, \quad a_4 = \gamma z_0^3, \quad b_0 = -\gamma y_0^3z_0^3, \quad b_1 = 2\gamma y_0^2z_0^3, \\ &(x_0y_0z_0)^2 + x_0y_0z_0 + 1 = 0.\end{aligned}$$

Let us set $\rho = x_0y_0z_0$; then $\rho^2 + \rho + 1 = 0$ and ω is of type

$$\begin{aligned}\omega &= \gamma\Big(x\mathrm{d}y - y\mathrm{d}x\Big)\Big(1 + 2y_0z_0(\rho+1)x - 2z_0y\Big) + 2\gamma z_0^2y^2\Big(y_0(\rho+2)x - y\Big)\mathrm{d}x + \gamma z_0^3y^3\Big(y - 2y_0x\Big)\mathrm{d}x \\ &\quad + 2\gamma y_0z_0^2x^2\Big(y_0\rho x - (2\rho+1)y\Big)\mathrm{d}y + \gamma y_0^2z_0^3x^3\Big(2y - y_0x\Big)\mathrm{d}y.\end{aligned}$$

This 1-form is linearly conjugated to

$$\omega_H^4 = (2x^3 - y^3 - 1)y\mathrm{d}x + (2y^3 - x^3 - 1)x\mathrm{d}y\,.$$

Indeed, the fact that ρ satisfies $\rho^2 + \rho + 1 = 0$ implies that

$$\omega_H^4 = \frac{9y_0z_0^2}{\gamma(\rho-1)}\varphi^*\omega, \quad \text{where } \varphi = \left(\frac{2\rho + 1 - (\rho+2)x - (\rho+2)y}{3y_0z_0}, \frac{(\rho-1)x - (2\rho+1)y + \rho + 2}{3z_0}\right).$$

□

We thank the anonymous referee for making us the following observation concerning the end of the proof of Theorem B.

Remark 3.5. — Once one shows that a degree 4 foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ with 12 invariant lines is such that each line contains 3 radial singularities of order 2, it is possible to show that the foliation is Hesse's pencil using an argument more geometric than the lengthy computations carried out before. Indeed let ℓ_0 be one of the invariant lines. Let ℓ_1 and ℓ_2 be the other invariant lines intersecting ℓ_0 at the two non radial singularities of $\mathcal{F}$ over ℓ_0. We claim that the intersection $p_0 = \ell_1 \cap \ell_2$ is a non radial singularity of $\mathcal{F}$. Aiming a contradiction assume this is not the case. As we have seen before, lines through two radial singular points of order 2 must be invariant, therefore the lines joining p_0 and $\ell_0 \cap \mathrm{Sing}(\mathcal{F}) = \{p_1, p_2, r_1, r_2, r_3\}$ are all invariant. Therefore p_0

must be a radial singular point of order 4: contradiction. It follows that we can divide the 12 invariant lines of $\mathcal{F}$ into 4 triangles ($\ell_0 \cup \ell_1 \cup \ell_2$ is one of them) such that any two triangles have intersection equal to the 9 radial singular points of $\mathcal{F}$ (the non radial singularities are the vertices). There is a pencil $\mathcal{P}$ of cubics containing these four triangles, which have 12 lines in common with the foliation $\mathcal{F}$. It turns out that $\mathcal{P} = \mathcal{F}$ because otherwise $12 \leq \deg \operatorname{Tang}(\mathcal{F}, \mathcal{P}) = \deg \mathcal{F} + \deg \mathcal{P} + 1 = 9$ (*see* for instance [12, Proposition 1.3.2]). Since two triangles intersect transversely, the general element of the pencil is smooth. Now, if we take a smooth cubic $\mathcal{C} = \{f(x,y,z) = 0\}$ in the pencil then its Hessian $\{\det(\operatorname{Hess}(f)) = 0\}$ is another cubic which intersect $\mathcal{C}$ at its inflection points. But since $\mathcal{F}$ is a convex foliation these points must be the 9 radial singularities of $\mathcal{F}$, *i.e.* the base points of the pencil. Hence $\mathcal{F}$ is the pencil defined by a smooth cubic and its Hessian. Thus $\mathcal{F}$ is tangent to the Hesse pencil according to [1, Section 2].

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